EXPONENTIAL DECAY OF THE SOLUTIONS OF QUASILINEAR SECOND-ORDER EQUATIONS AND POHOZAEV IDENTITIES

BY

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1. Introduction.

This paper is devoted to a rather detailed study of the pointwise exponential decay of the solutions of quasilinear second order equations on $\mathbb{R}^N$

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x,u,\nabla u)\partial_\alpha\partial_\beta u + b(x,u,\nabla u) = f, \tag{1.1}$$

under general assumptions about the coefficients $a_{\alpha\beta}$ and b and provided that the right-hand side f exhibits appropriate exponential decay at infinity. Furthermore, although our exposition is made in the case of $\mathbb{R}^N$ for simplicity, all of our results extend to more general unbounded domains. Some extensions are even valid in arbitrary unbounded domains, but the precise discussion of others would require going into geometric considerations and

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additional technicalities. For that reason, we have chosen to stay with the $\mathbb{R}^N$ setting and to mention these generalizations in various remarks.

In spite of a vast literature discussing the asymptotic properties of the solutions to all sorts of PDEs on a variety of unbounded domains, there is little that addresses the exponential decay issue (especially pointwise) in elliptic problems on the entire space. Much more is available for the case of cylindrical domains, see e.g. the recent book by Oleinik [13] and the references therein. Under severe restrictions about the structure of the operator in (1.1) and/or the nature of the solutions (e.g. positive solutions), some exponential decay results when $f = 0$ have been obtained in Berestycki and Lions [2], Lions [12], Rother [18], among others. It should be stressed that the exponential decay properties have usually been studied along with the existence question, which naturally places a few extra hurdles in the way of generality.

In some instances, exponentially decaying solutions have been found in appropriately weighted Sobolev spaces, as in Chaljub and Volkmann [4]. This does not give a pointwise result and leaves open the question about the asymptotic behavior of other decaying solutions. In addition, the exponential decay obtained this way is often suboptimal. In fact, by showing that "all" the decaying solutions must tend to 0 exponentially (see below), our work gives added legitimacy to this kind of approach as regards the existence issue.

When (1.1) is the linear equation $-\Delta u + V(x)u + \lambda u = 0$, i.e. the eigenvalue problem for a linear Schrödinger operator on $\mathbb{R}^N$, the exponential decay question has been thoroughly investigated, with Agmon's book [1] being one of the most acknowledged contributions. Yet, we have found no evidence that the asymptotic properties of the null functions of general linear equations with variable coefficients has been investigated. Such properties are obtained here as a special case.

In this work, we study the exponential decay of the solutions of (1.1) independently of their existence and we also examine the behavior of their first and second derivatives. As regards the solutions themselves, we prove that every solution in $C^1(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ tending uniformly to 0 at infinity along with its gradient must tend to 0 exponentially. This is even true for $W^{2,N}_{\mathrm{loc}}$ (hence C^0) solutions tending uniformly to 0 at infinity, with no assumption made about their gradients, when the coefficients $a_{\alpha\beta}$ are independent of

∇u and when b depends linearly upon ∇u.

Not all decaying solutions of elliptic (even linear elliptic) problems decay exponentially. Two main conditions are needed. First, a weak ellipticity-like requirement, roughly speaking "localized" along $\nabla u = 0$, which even allows for degenerate problems involving p-Laplacians or other singularities. The second main assumption is that $\partial_u b(x, 0)$ is positive and remains bounded away from 0 for large $|x|$. When this assumption fails, the decay need not be exponential, even when $f = 0$, see e.g. Véron [20] or Kametaka and Oleinik [10]. There are other assumptions, notably boundedness and equicontinuity conditions but no restriction is placed upon the growth of the coefficients $a_{\alpha\beta}$ and b with respect to u or ∇u. While the above presupposes that $\partial_u b(x, 0)$ exists, the nature of our assumptions enables us to handle some problems with even less smoothness available, as we show on one example.

The exponential decay of the solutions is studied in Section 2, where the issue is eventually settled via the maximum principle, which gives a good estimate for the rate of decay. The main result of that section, Theorem 2.1, is given in as much generality as we have been able to obtain. That makes the proof rather lengthy and delicate, but the full strength of the theorem is needed to incorporate "interesting" degenerate elliptic problems. In a somewhat disguised form and in a special case, the line of argument first appears in Jeanjean, Lucia and Stuart [8], where it serves a different purpose (properness). As we note in Remark 2.4, the results of Section 2 remain true, with the same proof, in general unbounded domains $\Omega \subset \mathbb{R}^N$ for solutions u with Supp $u_{|\partial\Omega}$ compact.

The value of the maximum principle in establishing the exponential decay of the solutions to special elliptic problems in special domains has a fairly long history, summarized in the survey paper by Horgan [7]. However, it does not appear that the all-around validity of the exponential decay phenomenon has been demonstrated before.

In Section 2, the solutions in $W^{2,p}(\mathbb{R}^N)$ for some $p \in (N, \infty)$ are of special interest, for they satisfy both the prerequisites of being in $C^1(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ and of tending uniformly to 0 at infinity along with their gradients. In general, the exponential decay of a solution to (1.1) in no way implies a similar property for its derivatives. For example, with $N = 1, u(x) = (\cosh\, x)^{-1} \cos(\sinh\, x)$ solves $-(\cosh\, x)^{-3} u'' - \tanh\, x (\cosh\, x)^{-3} u' + u =$

$[(\cosh x)^{-1} + (\cosh x)^{-2} + (\cosh x)^{-6}] \cos(\sinh x)$, an equation satisfying all the conditions required in Section 2, but no derivative of u tends to 0 at infinity. As we prove in Section 4, the situation is different for solutions $u \in W^{2,p}(\mathbb{R}^N), p \in (N, \infty)$, and the derivatives $D^\kappa u, |\kappa| \leq 2$, also decay exponentially under slightly more restrictive but still "localized" ellipticity requirements, mild smoothness assumptions (but p-Laplacians are now ruled out) and appropriate decay of f at infinity. The procedure hinges on the results of Section 2 and a rather careful evaluation of the constants arising in the Hölder estimates over balls with fixed radius but arbitrary center. Some technical material, developed in Section 3, is also involved.

As a by-product, we obtain that the $W^{2,p}$ solutions of (1.1) are "eventually" of class C^2 and in $W^{2,q}$ for every $1 \leq q \leq \infty$, i.e. this is true only in the complement of some closed ball depending upon the solution, unless more ellipticity is available. The exponential decay of the derivatives is more delicate to extend to general unbounded domains, and in fact this extension requires introducing some geometric limitations. This is loosely discussed, with no technical details, in Remark 4.4.

The exponential decay of the derivatives of the solutions up to order 2 suggests that identities of Pohozaev type [14] should be valid for such solutions corresponding to $f = 0$. This problem is addressed in Section 5 and the usual application to the nonexistence of nonzero solutions is treated in Section 6.

The identities worked out in Section 5 are in the same spirit as, but have noticeable differences with, those that can be found in Willem [21] or Kavian [11]. To avoid having to introduce a tedious list of technical assumptions, our exposition is confined to problems with x-independent coefficients and, of course, in a divergence form, as is customary in these matters. Once again, the ellipticity assumptions are rather weak.

The nonexistence theorems of Section 6 are consistent with those that should be expected from the results of Pucci and Serrin [15], not directly applicable here since the solutions of interest to us are merely in $W^{2,p}(\mathbb{R}^N)$ for some $p \in (N, \infty)$ but not of class C^2. A nonnegligible part of the work is to prove that the solutions satisfy an equation of the form (1.1), hence have the decay properties proved in Section 4 (and are C^2, but only in the complement of a ball depending on the solution). Once this is shown, the effort needed to establish the validity of the Pohozaev identities does not significantly exceed what it takes to prove that some boundary integral tends to 0, as required in [15], yet provides a somewhat stronger property. (In [15], the nonexistence of nonzero solutions on unbounded domains is established from identities valid on bounded domains; no identity on unbounded domains is proved or used.)

Remark 1.1: In recent work [17], we have shown that the nonexistence of nonzero solutions for (1.1) when the coefficients are x-independent or, more generally, N-periodic in x, is intimately related to the properness properties of the operator (from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N), p \in (N,\infty)$) represented by the left-hand side of (1.1). This relationship even extends to other (related) operators with x-dependent coefficients. This is to say that these nonexistence results in the x-independent coefficients case have a profound and far-reaching impact on much broader issues. This is true for more general domains and other problems (systems) as well. See for instance Galdi and Rabier [5] for the Navier-Stokes system on planar exterior domains. For the outcome of the existence of nonzero solutions, see Rabier [16]. $\square$

Most of the notation used throughout the article is standard, including $|\cdot|_{k,p,\Omega}$ and $||\cdot||_{m,p,\Omega}$ for the seminorms and norms in $W^{m,p}(\Omega), k \leq m$. We shall also make use of the spaces $C^{k,\sigma}(\Omega)$ and $C^{k,\sigma}(\overline{\Omega}), k \in \mathbb{N}, 0 < \sigma \leq 1$, the latter when Ω is a bounded open subset of $\mathbb{R}^N$. We use the notation

$$[u]_{0,\sigma,\overline{\Omega}} := \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\sigma} \tag{1.2}$$

for $u \in C^{0,\sigma}(\overline{\Omega})$ and

$$[u]_{k,\sigma,\overline{\Omega}} := \sum_{|\kappa|=k} [D^\kappa u]_{0,\sigma;\overline{\Omega}}\ , \tag{1.3}$$

for $u \in C^{k,\sigma}(\overline{\Omega}), k \in \mathbb{N}$. The norm on $C^{k,\sigma}(\overline{\Omega})$ is denoted by $[[\cdot]]_{k,\sigma,\overline{\Omega}}$. Precisely,

$$[[u]]_{k,\sigma,\overline{\Omega}} := ||u||_{k,\infty,\Omega} + [u]_{k,\sigma,\overline{\Omega}}\ . \tag{1.4}$$

If $g : \mathbb{R}^N \to \mathbb{R}$ is a (Lebesgue) measurable function, we set

$$\overline{\lim}_{|x|\to\infty}\, g(x) := \lim_{R\to\infty}\ \operatorname{ess}\ \sup_{\tilde{B}_R}\ g, \tag{1.5}$$

$$\underline{\lim}_{|x|\to\infty}\, g(x) := \lim_{R\to\infty}\ \operatorname{ess}\ \inf_{\tilde{B}_R}\ g, \tag{1.6}$$

where $\tilde{B}_R \subset \mathbb{R}^N$ is the complement of the closed ball of center 0 and radius R.

An important role is here played by the functions in $C^k(\mathbb{R}^N), k \in \mathbb{N}$, that tend to 0 at infinity along with their derivatives of order up to k. We denote this space by $C^k_d(\mathbb{R}^N)$, where "d" stands for "decay". No specific rate of decay is incorporated in this definition:

$$C^k_d(\mathbb{R}^N) := \{u \in C^k(\mathbb{R}^N) : \lim_{|x|\to\infty} D^\kappa u(x) = 0,\ \forall\, |\kappa| \leq k\}. \tag{1.7}$$

The space $C^k_d(\mathbb{R}^N)$ is a closed subspace of $W^{k,\infty}(\mathbb{R}^N)$.

Every mapping $g(= g(x,\xi)) : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ can be viewed as the bundle morphism

$$(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N+1} \longmapsto (x, g(x,\xi)) \in \mathbb{R}^N \times \mathbb{R}$$

between the (trivial) bundles $\mathbb{R}^N \times \mathbb{R}^{N+1}$ and $\mathbb{R}^N \times \mathbb{R}$, with base $\mathbb{R}^N$. The triviality of these bundles preempts the relevance of any result from vector bundle theory, but the terminology "bundle map" will turn out to be a useful one since it stresses a difference between the "base" variable x and the "fiber" variable ξ. This difference is important in a few places, beginning with

Definition 1.1. *The mapping $g : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ is said to be a C^0 bundle map equicontinuous at $\xi = 0$ if it is continuous and the collection $(g(x,\cdot))_{x\in\mathbb{R}^N}$ is equicontinuous at $\xi = 0$, and a C^0 equicontinuous bundle map if $(g(x,\cdot))_{x\in\mathbb{R}^N}$ is equicontinuous at every point of $\mathbb{R}^{N+1}$. More generally, if $k \in \mathbb{N}, g$ is called an equicontinuous C^k_ξ bundle map if $D^\kappa_\xi g$ exists and is an equicontinuous C^0 bundle map for $|\kappa| \leq k$.*

Lastly, the following rule is used throughout: when $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N+1}$ and the components of x and ξ must be displayed, we always write $x = (x_1, \cdots, x_N)$ and $\xi = (\xi_0, \cdots, \xi_N)$. Thus, when ξ is replaced by $(u(x), \nabla u(x)), u(x)$ occupies the slot of ξ_0 and $\partial_\alpha u(x)$ that of $\xi_\alpha, 1 \leq \alpha \leq N$.

2. Exponential decay of the solutions.

Consider the quasilinear equation on $\mathbb{R}^N$

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x,u,\nabla u)\partial_\alpha\partial_\beta u + b(x,u,\nabla u) = f. \tag{2.1}$$

As announced in the Introduction, we shall concern ourselves with the exponential decay of the solutions u of (2.1) under general hypotheses about the coefficients $a_{\alpha\beta}$ and b and, of course, under the assumption that the right-hand side f itself exhibits exponential decay at infinity. We henceforth denote by $A(x,\xi)$ the $N\times N$ symmetric matrix

$$A(x,\xi) := (a_{\alpha\beta}(x,\xi)), \tag{2.2}$$

and set

$$D^*(x,\xi) := |\det A(x,\xi)|^{1/N}. \tag{2.3}$$

The main assumptions are as follows. For $1 \le \alpha, \beta \le N$,

$$a_{\alpha\beta} = a_{\beta\alpha} \text{ is a } C^0 \text{ bundle map equicontinuous at } \xi = 0, \tag{2.4}$$

$$a_{\alpha\beta}(\cdot, 0) \in L^\infty(\mathbb{R}^N), \tag{2.5}$$

and the following ellipticity-like condition holds:

$$\begin{cases} \text{For every solution } u \in C^1_d(\mathbb{R}^N) \cap W^{2,N}_{\text{loc}}(\mathbb{R}^N) \text{ of (2.1), there is } R > 0 \\ \text{such that } A(\cdot, u, \nabla u) \text{ is positive definite a.e. on the set } \tilde{B}_R \setminus u^{-1}(0). \end{cases} \tag{2.6}$$

As regards condition (2.6), recall the definition of $C^1_d(\mathbb{R}^N)$ in (1.7) used throughout this article and that $\tilde{B}_R$ denotes the complement of $\overline{B}(0,R)$. Condition (2.6) holds when

$$\begin{cases} \text{there are } \epsilon > 0 \text{ and } R > 0 \text{ such that } A(x,\xi) \text{ is positive} \\ \text{definite for } |x| \ge R \text{ and } |\xi| \le \epsilon, \end{cases} \tag{2.6$'$}$$

and hence (2.6) is a rather weak ellipticity condition. The choice $a_{\alpha\beta}(x,\xi) = a_{\alpha\beta}(\xi) = \xi_0\delta_{\alpha\beta}$ (Kronecker delta), corresponding to the operator $u\Delta u$, gives an example satisfying (2.6) but not (2.6′). As we shall see in Example 2 later, a much more subtle argument reveals that condition (2.6) is also relevant in some problems involving p-Laplacians.

As for the coefficient b in (2.1), we shall assume that

$$b(x,\xi) = \sum_{j=0}^{N} c_j(x,\xi)\xi_j, \tag{2.7}$$

where, for $0 \leq j \leq N$,

$$c_j \text{ is a } C^0 \text{ bundle map equicontinuous at } \xi = 0, \tag{2.8}$$

$$c_j(\cdot,0) \in L^\infty(\mathbb{R}^N). \tag{2.9}$$

As a complement to condition (2.6), we shall require that:

$$\begin{cases} \text{For every solution } u \in C^1_d(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N) \text{ of (2.1), there is } R > 0 \\ \text{such that } D^*(\cdot,u,\nabla u) \neq 0 \text{ a.e. on } \tilde{B}_R \setminus u^{-1}(0) \text{ and } c_\alpha(\cdot,u,\nabla u)/D^*(\cdot,u,\nabla u) \in \\ L^N_{\mathrm{loc}}(\tilde{B}_R \setminus u^{-1}(0)), \quad 1 \leq \alpha \leq N, \end{cases} \tag{2.10}$$

where D^* is given by (2.3). Note that (2.10) does not involve c_0 and that it is trivially satisfied when (2.6) holds and $c_\alpha = 0, 1 \leq \alpha \leq N$, so that $c_\alpha(\cdot,u,\nabla u)/D^*(\cdot,u,\nabla u) = 0$ in $\tilde{B}_R$ for $R > 0$ large enough. Condition (2.10) is also vacuous when condition (2.6′) holds since $c_\alpha(\cdot,u,\nabla u)/D^*(\cdot,u,\nabla u) \in C^0(\tilde{B}_R) \subset L^N_{\mathrm{loc}}(\tilde{B}_R)$ for every $u \in C^1_d(\mathbb{R}^N)$ provided that $R > 0$ is large enough.

Lastly, we shall assume

$$0 < \delta^\infty := \underline{\lim}_{|x|\to\infty} c_0(x,0) \ (< \infty \text{ by (2.9)}). \tag{2.11}$$

With $\rho(x) =$ spectral radius of $A(x,0)$ (see (2.2)), we set

$$\rho^\infty := \overline{\lim}_{|x|\to\infty} \rho(x) \ (0 \leq \rho^\infty < \infty \text{ by (2.5)}) \tag{2.12}$$

$$c^\infty := \overline{\lim}_{|x|\to\infty}\{\sum_{\alpha=1}^{N} c_\alpha(x,0)^2\}^{1/2} \ (0 \leq c^\infty < \infty \text{ by (2.9)}). \tag{2.13}$$

Remark 2.1: Conditions (2.7) to (2.9) hold when b is a C^1_ξ bundle map with $b(\cdot,0)=0$ and $\partial_{\xi_j} b(\cdot,0) \in L^\infty(\mathbb{R}^N), 0 \le j \le N$. In that case, $c_j(x,\xi) = \int_0^1 \partial_{\xi_j} b(x,t\xi)dt$. That c_j is an equicontinuous C^0 bundle map (hence equicontinuous at $\xi = 0$) follows from the same property holding for $\partial_{\xi_j} b$ and the remark that this implies that $(\partial_{\xi_j} b(x,\cdot))_{x\in\mathbb{R}^N}$ is uniformly equicontinuous on the compact subsets of $\mathbb{R}^{N+1}$. Clearly, δ^∞ in (2.11) is just $\delta^\infty = \underline{\lim}_{|x|\to\infty} \partial_{\xi_0} b(x,0)$. However, this case is not the only one relevant to the applications; see Section 5. $\square$

Theorem 2.1. *Retain assumptions (2.4) to (2.11) and let $0 < \mu^* \le \infty$ be defined by*

$$\frac{1}{\mu^*} = \frac{c^\infty}{2\delta^\infty} + \sqrt{\left(\frac{c^\infty}{2\delta^\infty}\right)^2 + \frac{\rho^\infty}{\delta^\infty}}, \tag{2.14}$$

so that $\mu^* = \infty$ if and only if $\rho^\infty = c^\infty = 0$. Let $u \in C^1_d(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution of (2.1) with $f \in L^N_{\mathrm{loc}}(\mathbb{R}^N)$. Suppose that $\overline{\lim}_{|x|\to\infty} e^{\nu|x|}|f(x)| < \infty$ for some $\nu > 0$. Then, for any $\mu < \min(\mu^*,\nu)$ we have

$$\lim_{|x|\to\infty} e^{\mu|x|}u(x) = 0. \tag{2.15}$$

Proof. It suffices to prove (2.15) when $0 < \mu < \min(\mu^*,\nu)$. Set

$$q_j(x) := c_j(x,u(x),\nabla u(x)), \quad 0 \le j \le N. \tag{2.16}$$

Since u is C^1, q_j is continuous. Furthermore, by the equicontinuity of $(c_j(x,\cdot))_{x\in\mathbb{R}^N}$ at $\xi = 0$ (see (2.8)) and since $\lim_{|x|\to\infty} |u(x)| + |\nabla u(x)| = 0$, it is easily checked that

$$\lim_{|x|\to\infty} q_j(x) - c_j(x,0) = 0, \quad 0 \le j \le N. \tag{2.17}$$

¿From (2.7) and (2.16), we have

$$b(x,u(x),\nabla u(x)) = q_0(x)u(x) + \sum_{\alpha=1}^N q_\alpha(x)\partial_\alpha u(x). \tag{2.18}$$

Set

$$p_{\alpha\beta}(x) := a_{\alpha\beta}(x, u(x), \nabla u(x)), \quad 1 \le \alpha, \beta \le N, \tag{2.19}$$

and let L be the second-order differential operator defined by

$$L := \sum_{\alpha,\beta=1}^{N} p_{\alpha\beta}(x)\partial_\alpha\partial_\beta - \sum_{\alpha=1}^{N} q_\alpha(x)\partial_\alpha. \tag{2.20}$$

By (2.1), (2.18) and (2.19), we have

$$Lu = q_0 u - f. \tag{2.21}$$

We now fix $\mu \in (0, \min(\mu^*, \nu))$ and set

$$v(x) = e^{-\mu|x|}.$$

By a simple calculation, we have, for $x \neq 0$

$$(Lv)(x) = \frac{\mu v(x)}{r}\left\{\frac{\mu}{r}\left(1 + \frac{1}{\mu r}\right)\sum_{\alpha,\beta=1}^{N} p_{\alpha\beta}(x)x_\alpha x_\beta - \sum_{\alpha=1}^{N} p_{\alpha\alpha}(x) + \sum_{\alpha=1}^{N} q_\alpha(x)x_\alpha\right\}, \tag{2.22}$$

where $r := |x|$.

By the definition of μ^* in (2.14) and since $0 < \mu < \mu^*$, we have $\rho^\infty\mu^2 + c^\infty\mu - \delta^\infty < 0$. This is trivial if $\mu^* = \infty$, for then $\rho^\infty = c^\infty = 0$, and if $\mu^* < \infty$, then μ^* is the only positive root of the polynomial $\rho^\infty\lambda^2 + c^\infty\lambda - \delta^\infty$ (even when $\rho^\infty = 0$). It follows that we may choose constants $\overline{\rho} > \rho^\infty, \overline{c} > c^\infty$ and $\underline{\delta} \in (0, \delta^\infty)$, all depending upon μ, such that

$$\overline{\rho}\mu^2 + \overline{c}\mu - \underline{\delta} < 0. \tag{2.23}$$

By (2.12), for any $\overline{\rho}' \in (\rho^\infty, \rho)$, there is $R > 0$ such that

$$\left(1 + \frac{1}{\mu r}\right)\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, 0)\eta_\alpha\eta_\beta \le \overline{\rho}'|\eta|^2, \quad \forall\ \eta \in \mathbb{R}^N, \tag{2.24}$$

provided that $|x| = r \geq R$. Using the equicontinuity of the $a_{\alpha\beta}$ at $\xi = 0$ (see (2.4)) and, once again, $\lim_{|x|\to\infty} |u(x)| + |\nabla u(x)| = 0$, it follows from (2.19) and (2.24) that $\left(1 + \frac{1}{\mu r}\right) \sum_{\alpha,\beta=1}^{N} p_{\alpha\beta}(x)\eta_\alpha\eta_\beta \leq \overline{\rho}|\eta|^2$ for all $\eta \in \mathbb{R}^N$ provided that $|x| = r \geq R$ and $R > 0$ is large enough. In particular, by letting $\eta = x$, we get $\left(1 + \frac{1}{\mu r}\right) \sum_{\alpha,\beta=1}^{N} p_{\alpha\beta}(x)x_\alpha x_\beta \leq \overline{\rho} r^2$ for $|x| = r \geq R$. Together with (2.22), this yields

$$(Lv)(x) \leq v(x) \; [\overline{\rho}\mu^2 + |q(x)|\mu - \frac{\mu}{r} \sum_{\alpha=1}^{N} p_{\alpha\alpha}(x)] \text{ for } \quad |x| = r \geq R, \tag{2.25}$$

where $q := (q_1, \cdots, q_N)$.

Going back to (2.17) and recalling that $\overline{c} > c^\infty$ and $\underline{\delta} < \delta^\infty$, we see from (2.13) and (2.11) that given $\underline{\delta}' \in (\underline{\delta}, \delta^\infty)$, we can also ensure that

$$|q(x)| \leq \overline{c} \quad \text{and} \quad q_0(x) \geq (\underline{\delta}' + \underline{\delta})/2 \;\text{ for } |x| \geq R \tag{2.26}$$

by choosing $R > 0$ large enough. In addition, since $\mu < \nu$ and $\overline{\lim}_{|x|\to\infty} e^{\nu|x|}|f(x)| < \infty$, it is not restrictive to assume that

$$\frac{2|f(x)|e^{\mu|x|}}{\underline{\delta}' - \underline{\delta}} < 1 \text{ a.e. for } |x| \geq R. \tag{2.27}$$

We now fix $R > 0$ such that (2.25), (2.26) and (2.27) hold along with (2.6) for the solution u of interest. Choose $t \geq 1$ such that

$$u(x) < tv(x) \; (= te^{-\mu R}) \quad \text{whenever } |x| = R. \tag{2.28}$$

Observe that since $t \geq 1$ and $v(x) = e^{-\mu|x|}$, (2.27) implies

$$|f(x)| < tv(x)\frac{(\underline{\delta}' - \underline{\delta})}{2} \text{ a.e. for } |x| \geq R. \tag{2.29}$$

The next step of the proof consists of using the maximum principle to show that $u(x) \leq tv(x)$ for all x with $|x| \geq R$. In this aim, we set

$$m := \sup_{|x|\geq R} u(x) - tv(x). \tag{2.30}$$

Since both u and v are continuous and tend uniformly to 0 at infinity, we have that $m < \infty$. Also, assuming by contradiction that $m > 0$, the decay of u and v at infinity ensures the existence of $R^* > R$ such that $u(x) - tv(x) \leq m/2$ whenever $|x| \geq R^*$. The set

$$\omega(R, R^*) := \{x \in \mathbb{R}^N : R < |x| < R^*,\ u(x) > tv(x)\} \tag{2.31}$$

is open, bounded and nonempty (the latter by the positivity of m and the choice of R^*). Observe that if $x \in \partial\omega(R, R^*)$, then either $|x| = R^*$ or $u(x) = tv(x)$. Indeed, the option $|x| = R$ is ruled out by (2.28). This implies at once that $\overline{\omega}(R, R^*) \subset \tilde{B}_R$ and that

$$\max_{x \in \partial\omega(R,R^*)} u(x) - tv(x) \leq m/2. \tag{2.32}$$

By (2.32), m is not achieved on $\partial\omega(R, R^*)$. Since $u(x) - tv(x) \leq m/2$ for $|x| \geq R^*$, it follows from (2.31) that

$$m = \max_{x \in \omega(R,R^*)} u(x) - tv(x). \tag{2.33}$$

By the definition of $\omega(R, R^*)$ in (2.31), we have $u(x) > 0$ for $x \in \overline{\omega}(R, R^*)$. As a result, $\overline{\omega}(R, R^*) \subset \tilde{B}_R \setminus u^{-1}(0)$ and condition (2.6) implies that the matrix $A(\cdot, u, \nabla u)$ is positive definite a.e. on $\omega(R, R^*)$. Equivalently, the operator L is elliptic a.e. on $\omega(R, R^*)$. Since u is C^1 and A is continuous and symmetric, $A(\cdot, u, \nabla u)$ is positive semidefinite on $\omega(R, R^*)$. In particular, $p_{\alpha\alpha}(x) \geq 0$ in $\omega(R, R^*), 1 \leq \alpha \leq N$. By (2.25) it follows that

$$(Lv) \leq v(\underline{\rho}\mu^2 + |q|\mu) \quad \text{on } \omega(R, R^*). \tag{2.34}$$

By (2.26), $|q| \leq \overline{c}$ on $x \in \omega(R, R^*)$, whence $\overline{\rho}\mu^2 + |q|\mu \leq \overline{\rho}\mu^2 + \overline{c}\mu$, and since $\overline{\rho}\mu^2 + \overline{c}\mu < \underline{\delta}$ by (2.23), we infer from (2.34) that

$$(Lv) \leq \underline{\delta}v, \quad \text{on } \omega(R, R^*). \tag{2.35}$$

We have $[L(u - tv)] = (Lu) - t(Lv) = -f + q_0 u - t(Lv)$, a.e. on $\mathbb{R}^N$ where (2.21) was used. On $\omega(R, R^*)$, we also have that $q_0 \geq (\underline{\delta}' + \underline{\delta})/2$ (see (2.26)), $u > tv$ (see (2.31)) and

$(Lv) \leq \underline{\delta} v$ (see (2.35)). We thus find that $L(u - tv) \geq -f + t(\underline{\delta}' - \underline{\delta})v/2$ a.e. in $\omega(R, R^*)$, and (2.29) now yields

$$L(u - tv) > 0 \text{ a.e. on } \omega(R, R^*). \tag{2.36}$$

Since $u \in C^1(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ and the operator L is elliptic a.e. on $\omega(R, R^*)$, the maximum principle, e.g. in the form of [6, Theorem 9.1, p. 220], is applicable. In this respect, note that $u \in C^1(\overline{\omega}(R, R^*))$, whence the coefficients of L are in $L^\infty(\omega(R, R^*))$. The condition $|q|/|\det(p_{\alpha\beta})|^{1/N} \in L^N(\omega(R, R^*))$, needed for the validity of that theorem, follows from condition (2.10) and from $\overline{\omega}(R, R^*) \subset \tilde{B}_R \setminus u^{-1}(0)$. Last but not least, Theorem 9.1 of [6] is stated under the assumption that L is elliptic but nothing more than ellipticity a.e. is used in its proof. It thus follows from (2.36) that $\sup\limits_{x \in \omega(R, R^*)} u(x) - tv(x) = \max\limits_{x \in \partial\omega(R, R^*)} u(x) - tv(x)$, i.e. $m \leq m/2$ by (2.32) and (2.33). Thus, $m \leq 0$, in contradiction with the hypothesis $m > 0$. This shows that $m \leq 0$, i.e. $u(x) \leq tv(x)$ for $|x| \geq R$.

The equation (2.1) can be made into an equation for $-u$ by changing $a_{\alpha\beta}(x, \xi)$ into $a_{\alpha\beta}(x, -\xi), b(x, \xi)$ into $-b(x, -\xi)$ and $f(x)$ into $-f(x)$. The assumptions (2.2) to (2.11) are not affected by such a change. As a result, there are $R' > 0$ and $t' > 0$ such that $-u(x) \leq t'v(x)$ for $|x| \geq R'$. Thus, altogether, $|u(x)| \leq \max(t, t')v(x)$ for $|x| \geq \max(R, R')$. In other words, $e^{\mu|x|}u(x)$ is bounded on $\mathbb{R}^N$. By replacing μ by μ' with $\mu < \mu' < \min(\mu^*, \nu)$, we see that $\lim_{|x|\to\infty} e^{\mu|x|}u(x) = 0$, as was to be proved. $\square$

Since $W^{2,p}(\mathbb{R}^N) \hookrightarrow C^1_d(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ if $p \in (N, \infty)$, we obtain:

Corollary 2.1. *If $p \in (N, \infty)$, Theorem 2.1 holds for the solutions $u \in W^{2,p}(\mathbb{R}^N)$ of (2.1).*

In Corollary 2.1, the condition $f \in L^p(\mathbb{R}^N)$ is necessary for consistency.

Remark 2.2: Theorem 2.1 and Corollary 2.1 can be generalized further: (i) The equicontinuity requirement in (2.8) can be weakened, which is especially important regarding the coefficient c_0. Specifically, the proof of Theorem 2.1 goes through if the function $c_0(x, \xi)\xi_0$ is continuous in (x, ξ) (which does not even require $c_0(x, \xi)$ to be defined for $\xi_0 = 0$) and $\delta^\infty := \underline{\lim}_{|x|\to\infty, |\xi|\to 0, \xi_0 \neq 0} c_0(x, \xi) > 0$. This allows for $\delta^\infty = \infty$. The only modification consists in adding the provision that $u(x) \neq 0$ in the second relation (2.26), which indeed is used only in that case. A relevant example is given by $b(x,\xi) = |\xi_0|^{t-1}\xi_0$ with $t \in (0,1)$, for which $c_0(x,\xi) = |\xi_0|^{t-1}$. (ii) Even when conditions (2.5) and (2.10) do not hold for all the solutions in $C^1_d(\mathbb{R}^N) \cap W^{2,N}(\mathbb{R}^N)$, Theorem 2.1 remains valid for the solutions that comply with those conditions. (iii) If attention is confined to positive solutions, the exponential decay of f may be replaced by the exponential decay of f_+, the positive part of f, and conditions (2.6) and (2.10) must hold only for positive solutions. $\square$

Example 1: The equation ($N = 1$) $-u'' + u - u^3 = 0$ has the nonzero solution $u(x) = \sqrt{2}/\cosh x$ for which $\lim_{|x|\to\infty} e^{\mu|x|}u(x) = 0$ if and only if $\mu < 1$. Here, $\delta^\infty = \rho^\infty = 1, c^\infty = 0$, whence $\mu^* = 1$. Since $f = 0$, we have $\min(\mu^*, \nu) = \mu^* = 1$, i.e. the conclusion given by Theorem 2.1 is optimal for this example.

It is obvious how to construct N-dimensional examples satisfying condition (2.6′) (and hence (2.10) is satisfied vacuously). Instead of discussing such examples, we now show the relevance of conditions (2.6) and (2.10) in a nontrivial setting.

Example 2: Consider the problem

$$-\nabla \cdot (|\nabla u|^s \nabla u) + \sum_{\alpha=1}^{N} c_\alpha(x,u,\nabla u)\partial_\alpha u + \varphi(x,u) = f, \tag{2.37}$$

where $s > 0, \varphi = \varphi(x,\xi_0) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is of class C^1 with $\varphi(\cdot,0) = 0$, $c_\alpha : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ is continuous, $1 \leq \alpha \leq N$, and $f \in L^N_{\text{loc}}(\mathbb{R}^N)$. For $u \in C^1(\mathbb{R}^N) \cap W^{2,N}_{\text{loc}}(\mathbb{R}^N)$, (2.37) may be rewritten as

$$-s \sum_{\alpha,\beta=1}^{N} |\nabla u|^{s-2}\partial_\alpha u \partial_\beta u \partial_\alpha \partial_\beta u - |\nabla u|^s \Delta u + \sum_{\alpha=1}^{N} c_\alpha(x,u,\nabla u)\partial_\alpha u + \varphi(x,u) = f. \tag{2.37′}$$

In this problem, $a_{\alpha\beta}(x,\xi) = a_{\alpha\beta}(\xi') := s|\xi'|^{s-2}\xi_\alpha\xi_\beta + |\xi'|^s\delta_{\alpha\beta}, 1 \leq \alpha, \beta \leq N$, where $\xi' := (\xi_1, \cdots, \xi_N)$ and $\delta_{\alpha\beta}$ is the Kronecker delta. Since $s > 0, a_{\alpha\beta}$ is continuous at $\xi' = 0$ with $a_{\alpha\beta}(0) = 0$. Also, $c_0(x,\xi) = c_0(x,\xi_0) = \varphi(x,\xi_0)/\xi_0$ if $\xi_0 \neq 0, c_0(x,0) = \partial_{\xi_0}\varphi(x,0)$.

Clearly, the matrix $A(\xi') = (a_{\alpha\beta}(\xi'))$ is not positive definite if and only if $\xi' = 0$, so that $A(\nabla u(x))$ is not positive definite exactly when $\nabla u(x) = 0$. At a first sight, this has nothing to do with condition (2.6). However, let $u \in C^1_d(\mathbb{R}^N) \cap W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ be a solution of (2.37), so that (2.37′) holds a.e. on $\mathbb{R}^N$, say pointwise on $\mathbb{R}^N \setminus Z_1$, where meas $Z_1 = 0$.

With no loss of generality, we may also assume that $\partial_\alpha\partial_\beta u(x) \in \mathbb{R}$ and $f(x) \in \mathbb{R}$ for $x \in \mathbb{R}^N \setminus Z_1, 1 \leq \alpha, \beta \leq N$. Thus, if $x \notin Z_1$ and $A(\nabla u(x))$ is not positive definite, it follows from (2.37′) that $\varphi(x, u(x)) - f(x) = 0$.

¿From now on, we strengthen the assumption $f \in L^N_{\mathrm{loc}}(\mathbb{R}^N)$ by requiring that, in addition, $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $1 \leq q \leq \infty$. Since both φ and u are C^1, we have $\varphi(\cdot, u) - f \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^N)$. Recall that if $w \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^N)$ and $k \in \mathbb{R}$, then $\nabla w = 0$ a.e. on $w^{-1}(k)$ (see Brézis [3, p. 195], or Stampacchia [19] for a full proof). Thus, there is a subset $Z_2 \subset \mathbb{R}^N$ with meas $Z_2 = 0$ such that $\nabla(\varphi(\cdot, u) - f)(x) = \nabla_x \varphi(x, u(x)) + \partial_{\xi_0}\varphi(x, u(x))\nabla u(x) - \nabla f(x) = 0$ whenever $x \notin Z_2$ and $\varphi(x, u(x)) - f(x) = 0$. As a result, if $x \notin Z_1 \cup Z_2$ and $A(\nabla u(x))$ is not positive definite, whence $\nabla u(x) = 0$, we have not only $\varphi(x, u(x)) - f(x) = 0$ but also $\nabla_x \varphi(x, u(x)) - \nabla f(x) = 0$.

We now introduce an extra condition about φ: We assume that there are $R > 0$ and $\xi_0^* > 0$ such that $\partial_{\xi_0}\varphi(x, \xi_0) > 0$ for $|x| > R$ and $|\xi_0| < \xi_0^*$. This implies the existence of a C^1 function $\psi = \psi(x, \eta_0)$ defined on an open neighborhood $\mathcal{V}_R$ of $\tilde{B}_R \times \{0\}$ in $\mathbb{R}^N \times \mathbb{R}$ such that $\{(x, \xi_0) \in \tilde{B}_R \times (-\xi_0^*, \xi_0^*), \varphi(x, \xi_0) = \eta_0\} \Leftrightarrow \{(x, \eta_0) \in \mathcal{V}_R, \xi_0 = \psi(x, \eta_0)\}$. Note that $\psi(\cdot, 0) = 0$ (because $\varphi(\cdot, 0) = 0$). Since $u \in C^1_d(\mathbb{R}^N)$, we have $|u(x)| < \xi_0^*$ for $|x| > R$ after increasing $R > 0$ if necessary. This shows that if $|x| > R$ and $\varphi(x, u(x)) - f(x) = 0$, then $u(x) = \psi(x, f(x))$.

In summary, if $x \notin Z_1 \cup Z_2, |x| > R$ and $A(\nabla u(x))$ is not positive definite, then $u(x) = \psi(x, f(x))$ and $\nabla_x \varphi(x, \psi(x, f(x)) - \nabla f(x) = 0$. By a simple calculation based on the identity $\varphi(x, \psi(x, \eta_0)) = \eta_0$ for $(x, \eta_0) \in \mathcal{V}_R$, we find $\nabla_x \varphi(x, \varphi(x, \eta_0)) = -\nabla_x \psi(x, \eta_0)/\partial_{\eta_0}\psi(x, \eta_0)$. By letting $\eta_0 = f(x)$, it follows that the condition $\nabla_x \varphi(x, \psi(x, f(x))) - \nabla f(x) = 0$ may be rewritten as $\nabla(\psi(\cdot, f))(x) = 0$. (This requires checking the validity of the chain rule for $\psi(\cdot, f)$ and possibly enlarging $Z_1 \cup Z_2$ by a set of measure 0; we skip the details.)

To obtain the result that condition (2.6) holds, we just need to introduce a final assumption about the functions ψ and f: We assume that there is a subset $Z_3 \subset \mathbb{R}^N$ with meas $Z_3 = 0$ such that

$$\{(x, f(x)) \in \tilde{\mathcal{V}}_R,\ x \notin Z_3, \nabla(\psi(\cdot, f))(x) = 0\} \Rightarrow f(x) = 0. \tag{2.38}$$

Due to (2.38) we see that if $x \notin Z_1 \cup Z_2 \cup Z_3, |x| > R$ and $A(\nabla u(x))$ is not positive definite, then, $f(x) = 0$, whence $u(x) = \psi(x, 0) = 0$, and (2.6) holds since meas $(Z_1 \cup Z_2 \cup Z_3) = 0$. Observe that all the ingredients in (2.6) have been used above, i.e. a less general variant of (2.6) would not suffice for the problem (2.37).

Condition (2.38) is trivially satisfied if $f = 0$ or, more generally, if f has compact support. Condition (2.38) also holds if $\nabla(\psi(\cdot, f))$ vanishes only on a set of measure 0 in $\tilde{B}_R$, a "generic" property, at least for smooth functions. This is to say that condition (2.38) is little restrictive in practice. It fails e.g. when $\psi(\cdot, f)$ is constant and nonzero on some nonempty open subset of $\mathbb{R}^N$.

Since the coefficients $a_{\alpha\beta}$ are here independent of x, conditions (2.4) and (2.5) are also vacuous. From Remark 2.1, conditions (2.7) to (2.8) hold for $j = 0$ if φ is a C^1_ξ bundle map with, in addition to $\varphi(\cdot, 0) = 0$, also $\partial_{\xi_0}\varphi(\cdot, 0) \in L^\infty(\mathbb{R}^N)$. For the other indices $j = \alpha, 1 \leq \alpha \leq N$, conditions (2.7) and (2.8) must be required of the coefficients c_α in (2.37). By a simple calculation, $D^*(x, \xi) = D^*(\xi') = (1 + s)^{1/N}|\xi'|^s$ (observe that the eigenvalues of $A(\xi')$ are $|\xi'|^s$, with multiplicity $N - 1$, and $(1 + s)|\xi'|^s$, with multiplicity 1). Thus, condition (2.10) holds if $c_\alpha(x, \xi_0, \xi')/|\xi'|^s$ is locally bounded, which of course requires $c_\alpha(x, \xi_0, 0) = 0$ since $s > 0$. Thus, $c^\infty = 0$ in (2.13). Condition (2.11) is simply $\underline{\lim}_{|x|\to\infty}\partial_{\xi_0}\varphi(x, 0) > 0$. Since $\rho^\infty = c^\infty = 0$ in (2.12) and (2.13), we have $\mu^* = \infty$ in Theorem 2.1.

In our third and last example, we discuss a variant of Example 2 with less regularity of the coefficients than what is needed for the direct application of Theorem 2.1.

Example 3: Consider the problem

$$-\nabla \cdot (|\nabla u|^s \nabla u) + |u|^{t-1}u + k(x)|u|^{\sigma-1}u = 0, \tag{2.39}$$

where $s \geq 0, t \in (0, 1), \sigma > t$ and k is of class C^1 and bounded. When $s > 0$, this is a special case of Example 2, except that $\varphi(x, \xi_0) := |\xi_0|^{t-1}\xi_0 + k(x)|\xi_0|^{\sigma-1}\xi_0$ is not of class C^1 because $t < 1$. In fact, for this example the coefficient $b = \varphi$ does not satisfy (2.7) - (2.8). However, every solution $u \in C^1_d(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ of (2.39) must solve the problem

$$-|u|^{1-t}\nabla \cdot (|\nabla u|^s \nabla u) + u + k(x)|u|^{\sigma-t}u = 0, \tag{2.40}$$

obtained by multiplying (2.39) by $|u|^{1-t}$. This problem fits in the general class

$$-|u|^r \nabla \cdot (|\nabla u|^s \nabla u) + \varphi(x, u) = 0, \tag{2.41}$$

with $r > 0$ and φ of class C^1. Condition (2.6) holds trivially if $s = 0$. If $s > 0$, (2.41) is more degenerate than (2.37), but the exact same conditions about φ as in Example 2 (with $f = 0$) ensure that (2.6) holds (and (2.10) is vacuous since $c_\alpha = 0, 1 \leq \alpha \leq N$). By the boundedness of k and $\sigma > t$, it is plain that $\varphi(x, \xi_0) := \xi_0 + k(x)|\xi_0|^{\sigma - t}\xi_0$ in (2.40) satisfies those conditions. When $s = t - 1$ (hence $s < 0$, a case not covered here) the exponential decay of the *entire* solutions of (2.39) has been studied by Kabeya [9].

In an important special case, Theorem 2.1 and Corollary 2.1 remain valid under milder assumptions about the solution u.

Theorem 2.2. *Suppose, in addition to the assumptions of Theorem 2.1, that the coefficients $a_{\alpha\beta}$ and $c_j, 1 \leq \alpha, \beta \leq N, 0 \leq j \leq N$, depend only upon x and ξ_0 and that conditions (2.6) and (2.10) hold for every solution $u \in C_d^0(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ of (2.1). Then, Theorem 2.1 is valid for every such solution.*

Proof. An inspection of the proof of Theorem 2.1 reveals that under the stated additional hypotheses, the continuity of ∇u and the condition $\lim_{|x| \to \infty} |\nabla u(x)| = 0$ become immaterial. $\square$

Corollary 2.2. *Theorem 2.2 holds for the solutions $u \in W^{1,p}(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ provided that $p \in (N, \infty)$*(1).

Theorem 2.2 and Corollary 2.2 are applicable to *linear* equations (2.1), i.e. equations of the form $Lu = f$ where L is the operator

$$L := -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{N} B_\alpha(x)\partial_\alpha + C(x), \tag{2.42}$$

with coefficents

$$A_{\alpha\beta} = A_{\beta\alpha}, B_\alpha, C \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \ 1 \leq \alpha, \beta \leq N. \tag{2.43}$$

(1)And (2.6) and (2.10) need to be checked only for those solutions.

Let $A(x) := (A_{\alpha\beta}(x))$ and assume that there is $R > 0$ such that

$$A \text{ is positive definite a.e. on } \tilde{B}_R, \tag{2.44}$$

$$B_\alpha / |\det A|^{1/N} \in L^N_{\mathrm{loc}}(\tilde{B}_R), 1 \leq \alpha \leq N. \tag{2.45}$$

Assume further that

$$0 < \delta^\infty := \underline{\lim}_{|x|\to\infty} C(x), \tag{2.46}$$

and set

$$\rho^\infty := \overline{\lim}_{|x|\to\infty} \rho(x) \quad (0 \leq \rho^\infty < \infty), \tag{2.47}$$

where $\rho(x)$ is the spectral radius of $A(x)$, and

$$c^\infty := \overline{\lim}_{|x|\to\infty} \{\sum_{\alpha=1}^{N} B_\alpha(x)^2\}^{1/2} \quad (0 \leq c^\infty < \infty). \tag{2.48}$$

Corollary 2.3. *Retain assumptions (2.43) to (2.46) and let $\mu^* > 0$ be defined by (2.14) with $\delta^\infty, \rho^\infty$ and c^∞ given by (2.46), (2.47) and (2.48), respectively. Let $u \in C^0_d(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution of $Lu = f$ with L given by (2.42) and $f \in L^N_{\mathrm{loc}}(\mathbb{R}^N)$. Suppose that there is $\nu > 0$ such that $\overline{\lim}_{|x|\to\infty} e^{\nu|x|}|f(x)| = 0$. Then, for any $\mu < \min(\mu^*, \nu)$ we have $\lim_{|x|\to\infty} e^{\mu|x|}u(x) = 0$. (In particular, this is true for the solutions $u \in W^{1,p}(\mathbb{R}^N) \cap W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N), p \in (N, \infty)$.)*

Proof. It is trivial to check that the conditions required in Theorem 2.2 or Corollary 2.2 are satisfied. Notice that the equicontinuity at $\xi = 0$ is trivial here since the coefficients are ξ-independent. $\square$

Remark 2.3: A direct proof of Corollary 2.3 reveals that the continuity of the coefficients of L can be omitted. In that form, the hypotheses made in Corollary 2.3 are almost exactly the same as those needed in [6, Theorem 9.1, p. 220] to ascertain that the operator L satisfies the maximum principle. The only difference is condition (2.46), slightly stronger than $C \geq 0$, which suffices in [6]. $\square$

The generalizations of Theorems 2.1 and Corollary 2.1 mentioned in Remark 2.2 remain valid in the setting of Theorem 2.2 and Corollaries 2.2 and 2.3.

As is customary, let us call generalized null-space of L in $C^0(\mathbb{R}^N) \cap W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ the set of vectors $u \in C^0(\mathbb{R}^N) \cap W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ for which there is some integer $n \in \mathbb{N}$ such that $L^k u \in C^0(\mathbb{R}^N) \cap W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ for $1 \leq k \leq n$ and $L^n u = 0$. From Corollary 2.3 and a straightforward induction argument, we obtain

Corollary 2.4. *Retain assumptions (2.43) to (2.46) and let $\mu^* > 0$ be defined by (2.14) with $\delta^\infty, \rho^\infty$ and c^∞ given by (2.46), (2.47) and (2.48), respectively. Let L denote the operator (2.42). If u is an element of the generalized null-space of L in $C^0(\mathbb{R}^N) \cap W^{2,N}_{\text{loc}}(\mathbb{R}^N)$ and if $\mu < \mu^*$, we have $\lim_{|x|\to\infty} e^{\mu|x|}u(x) = 0$.*

Remark 2.4: It is worth pointing out that all the results of this section extend to the case when $\mathbb{R}^N$ is replaced by an arbitrary unbounded open subset of $\mathbb{R}^N$ (with $\partial\Omega$ Lipschitz continuous in the setting of Corollaries 2.1, 2.2 and 2.3) and the solutions u are also continuous on $\overline{\Omega}$ with Supp $u_{|\partial\Omega}$ compact. The main argument, i.e. the application of the maximum principle in the proof of Theorem 2.1, can be repeated since the closure of the set $\omega(R, R^*)$, now a subset of Ω, contains no point of $\partial\Omega$. Indeed, $u(x) = 0$ if $x \in \partial\Omega$ and $|x| \geq R$ with R large enough, while $u \geq v > 0$ on $\overline{\omega}(R, R^*)$. Thus, it is still true that $x \in \partial\omega(R, R^*)$ implies either $|x| = R^*$ or $u(x) = tv(x)$, whence (2.32) continues to hold and the proof can be completed as previously. $\square$

3. Bundle maps of class $C^{0,s}$ semi-uniformly in the base variable.

This short section deals with technical material needed later. Recall that if $g = g(x, \xi)$ is a bundle map, x is referred to as the "base" variable.

Definition 3.1. *Let $0 < s \leq 1$ and let $g : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ be of class $C^{0,s}$. Considering $g = g(x, \xi)$ as a bundle map, we shall say that g is of class $C^{0,s}$ semi-uniformly in the base variable if for every $r > 0$ and every compact subset $K \subset \mathbb{R}^{N+1}$ there is a constant*

$C(r,K)>0$ such that $|g(x,\xi)-g(y,\zeta)| \leq C(r,K)[|x-y|^s+|\xi-\zeta|^s]$ whenever $|x-y|\leq 2r$ and $\xi,\zeta\in K$. If g is independent of ξ, we shall simply say that g is semi-uniformly of class $C^{0,s}$.

When $g=g(\xi)$ is independent of x, the concept introduced in Definition 3.1 is just the $C^{0,s}$ regularity of g. It is easily seen that the constant $C(r,K)$ in Definition 3.1 can always be chosen of the form $C(r,K)=C_0(K)r^{1-s}$, which shows that when $s=1$ (but not when $0<s<1$), "semi-uniformly" is the same as "uniformly". Although the proof of Lemma 3.1 below is trivial, hence omitted, "semi-uniformly" cannot be replaced by "uniformly".

Lemma 3.1. *Let $0<s'<s\leq 1$ and let $g:\mathbb{R}^N\times\mathbb{R}^{N+1}\to\mathbb{R}$ be a bundle map of class $C^{0,s}$ semi-uniformly in the base variable. Then, g is of class $C^{0,s'}$ semi-uniformly in the base variable.*

¿From Lemma 3.1, if g is of class C^1 with first derivatives bounded on all sets of the form $\mathbb{R}^N\times K$ with $K\subset\mathbb{R}^{N+1}$ compact, then g is of class $C^{0,s}$ semi-uniformly in the base variable for every $0<s\leq 1$. Evidently, the latter concept is more general and includes simple and natural examples which do not comply with the C^1 requirement.

Lemma 3.2. *Let $0<s\leq 1$ and let $g:\mathbb{R}^N\times\mathbb{R}^{N+1}\to\mathbb{R}$ be an equicontinuous C^0 bundle map. Suppose also that g is of class $C^{0,s}$ semi-uniformly in the base variable and that $g(\cdot,0)\in L^\infty(\mathbb{R}^N)$. Let $r>0$ be given and let $K\subset\mathbb{R}^{N+1}$ be a compact subset. There is a constant $D(r,K)>0$ such that, for every $0<\sigma\leq s$, for every mapping $w:\mathbb{R}^N\to\mathbb{R}^{N+1}$ of class $C^{0,\sigma/s}$ with $w(\mathbb{R}^N)\subset K$ and for every open ball $B_r\subset\mathbb{R}^N$ of radius r, we have*

$$[[g(\cdot,w)]]_{0,\sigma,\overline{B}_r} \leq D(r,K)[1+[w]^s_{0,\sigma/s,\overline{B}_r}]. \tag{3.1}$$

Proof. It is not very hard to prove (or see [17, Lemma 2.1]) that since g is an equicontinuous C^0 bundle map with $g(\cdot,0)\in L^\infty(\mathbb{R}^N)$, then the collection $(g(x,\cdot))_{x\in\mathbb{R}^N}$ is equibounded on the compact subsets of $\mathbb{R}^{N+1}$. Hence, there is a constant $D_0(K)>0$ such that $|g(x,\xi)|\leq D_0(K)$ for every $(x,\xi)\in\mathbb{R}^N\times K$. In particular,

$$|g(\cdot,w)|_{0,\infty,B_r} \leq D_0(K). \tag{3.2}$$

Let now $x, y \in B_r, x \neq y$. By Definition 3.1, $|g(x, w(x)) - g(y, w(y))|/|x - y|^\sigma \leq C(r, K)[|x - y|^{s-\sigma} + |w(x) - w(y)|^s/|x - y|^\sigma]$ and hence

$$(3.3) \qquad [g(\cdot, w)]_{0,\sigma,\overline{B}_r} \leq C(r, K)[(2r)^{s-\sigma} + [w]^s_{0,\sigma/s,\overline{B}_r}].$$

Since $0 \leq s - \sigma < 1$, (3.1) is obtained by adding up (3.2) and (3.3) and letting $D(r, K) := D_0(K) + \max(2r, 1)C(r, K)$. $\square$

Theorem 3.1. *Let $0 < s \leq 1$ and let $g : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ be an equicontinuous C^0 bundle map. Suppose also that g is of class $C^{0,s}$ semi-uniformly in the base variable and that $g(\cdot, 0) \in L^\infty(\mathbb{R}^N)$. Let $p \in (N, \infty), r > 0$ and $0 < \sigma \leq s(1 - N/p)$ be given.*
(i) If $u \in W^{2,p}(\mathbb{R}^N)$, then $g(\cdot, u, \nabla u) \in C^{0,\sigma}(\mathbb{R}^N)$ and

$$(3.4) \qquad \sup_{B_r} [[g(\cdot, u, \nabla u)]]_{0,\sigma,\overline{B}_r} < \infty,$$

where the supremum is taken over all the open balls $B_r \subset \mathbb{R}^N$ with radius r.
(ii) If g depends only upon x and ξ_0 and if $u \in W^{1,p}(\mathbb{R}^N)$, then $g(\cdot, u) \in C^{0,\sigma}(\mathbb{R}^N)$ and (3.4) holds, i.e.

$$(3.5) \qquad \sup_{B_r} [[g(\cdot, u)]]_{0,\sigma,\overline{B}_r} < \infty.$$

Proof. (i) It suffices to prove (3.4). From the choice of σ, we have $0 < \sigma/s \leq 1 - N/p$ and hence $W^{1,p}(B_r) \hookrightarrow C^{0,\sigma/s}(\overline{B}_r)$. Let $\theta(r)$ denote the embedding constant. By the translation invariance of the $W^{1,p}$ and $C^{0,\sigma/s}$ norms, $\theta(r)$ depends upon r (and also p and σ/s) but not upon the center of B_r.

Let $K \subset \mathbb{R}^{N+1}$ be a compact subset such that $(u(x), \nabla u(x)) \in K$ for every $x \in \mathbb{R}^N$ (recall $W^{2,p}(\mathbb{R}^N) \hookrightarrow C^1_d(\mathbb{R}^N)$ since $p > N$). By Lemma 3.2 with $w = (u, \nabla u)$, we have $[[g(\cdot, u, \nabla u)]]_{0,\sigma,\overline{B}_r} \leq D(r, K)[1 + [(u, \nabla u)]^s_{0,\sigma/s,\overline{B}_r}] \leq D(r, K)[1 + [[(u, \nabla u)]]^s_{0,\sigma/s,\overline{B}_r}] \leq D(r, K)[1 + \theta(r)^s ||u||^s_{2,p,B_r}]$ after possibly modifying $\theta(r)$ in a way depending only upon N. Since $||u||_{2,p,B_r} \leq ||u||_{2,p,\mathbb{R}^N}$ we obtain $[[g(\cdot, u, \nabla u)]]_{0,\sigma,\overline{B}_r} \leq D(r, K)[1 + \theta(r)^s ||u||^s_{2,p,\mathbb{R}^N}]$, and the right-hand side is independent of the center of B_r.

(ii) Modify the proof of (i) above in the obvious way. □

To complete this section, we show how to construct new mappings satisfying the conditions of Theorem 3.1 from old ones.

Theorem 3.2. *Let $0 < s \leq 1$ and let $g : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ be an equicontinuous C^0 bundle map. Suppose that g is of class $C^{0,s}$ semi-uniformly in the base variable and that $g(\cdot, 0) \in L^\infty(\mathbb{R}^N)$. Define $h : \mathbb{R}^N \times \mathbb{R}^{N+1} \to \mathbb{R}$ by $h(x, \xi) := \int_0^1 g(x, t\xi)dt$. Then, h is an equicontinuous C^0 bundle map, $h(\cdot, 0) \in L^\infty(\mathbb{R}^N)$ and h is of class $C^{0,s}$ semi-uniformly in the base variable.*

Proof. Obviously, $h(\cdot, 0) = g(\cdot, 0) \in L^\infty(\mathbb{R}^N)$. That h is an equicontinuous C^0 bundle map follows at once from the collection $(g(x, \cdot))_{x \in \mathbb{R}^N}$ being uniformly equicontinuous on the compact subsets of $\mathbb{R}^{N+1}$ ([17, Lemma 2.1]). Lastly, let $r > 0$ be fixed and let $K \subset \mathbb{R}^{N+1}$ be a compact subset. If $x, y \in \mathbb{R}^N, |x - y| \leq 2r$ and $\xi, \zeta \in K$, we have $|h(x, \xi) - h(y, \zeta)| \leq \int_0^1 |g(x, t\xi) - g(y, t\zeta)|dt$. Denote by $\tilde{K}$ the (compact) convex hull of $K \cup \{0\}$. Then $t\xi, t\zeta \in \tilde{K}$ for $t \in [0, 1]$ and hence, by Definition 3.1, $|g(x, t\xi) - g(y, t\zeta)| \leq C(r, \tilde{K})[|x-y|^s + t^s|\xi - \zeta|^s]$, so that $|h(x, \xi) - h(y, \zeta)| \leq C(r, \tilde{K})[|x-y|^s + (s+1)^{-1}|\xi - \zeta|^s] \leq C(r, \tilde{K})[|x-y|^s + |\xi - \zeta|^s]$. □

4. Exponential decay of the derivatives.

We shall now complement the results of Section 2 with an investigation of the asymptotic behavior of the first and second derivatives of the solutions $u \in W^{2,p}(\mathbb{R}^N)$ of (2.1) when $p \in (N, \infty)$ (and in some cases, $p \in (N/2, \infty), p > 1$). Doing so will require the introduction of stronger hypotheses ruling out some degenerate elliptic problems which were admissible in Section 2. Nevertheless, the ellipticity requirements remain very much "localized".

In what follows, $0 < s \leq 1$ denotes a real number which may a priori be different in different places. However, Lemma 3.1 implies that s may be chosen the same everywhere with no loss of generality.

As regards the coefficients $a_{\alpha\beta}$ in (2.1), we shall assume for $1 \leq \alpha, \beta \leq N$ that

$$a_{\alpha\beta} = a_{\beta\alpha} \text{ is an equicontinuous } C^0 \text{ bundle map,} \tag{4.1}$$

$$a_{\alpha\beta} \text{ is of class } C^{0,s} \text{ semi-uniformly in the base variable,} \tag{4.2}$$

$$a_{\alpha\beta}(\cdot, 0) \in L^\infty(\mathbb{R}^N), \tag{4.3}$$

$$\begin{cases} \text{there are constants } R > 0 \text{ and } \gamma > 0 \text{ such that} \\ \sum\limits_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, 0)\eta_\alpha\eta_\beta \geq \gamma|\eta|^2, \\ \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq R \text{ and all } \eta \in \mathbb{R}^N. \end{cases} \tag{4.4}$$

We shall use (4.4) in the following seemingly stronger form

Lemma 4.1. *Under conditions (4.1) and (4.4), there is $\epsilon > 0$ such that*

$$\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, \xi)\eta_\alpha\eta_\beta \geq \frac{\gamma}{2}|\eta|^2, \tag{4.4$'$}$$

for all $x \in \mathbb{R}^N$ with $|x| \geq R$, all $\xi \in \mathbb{R}^{N+1}$ with $|\xi| \leq \epsilon$ and all $\eta \in \mathbb{R}^N$.

Proof. This follows from (4.4) and the equicontinuity of $(a_{\alpha\beta}(x, \cdot))_{x \in \mathbb{R}^N}$ at $\xi = 0$. $\square$

For the coefficient b, we shall require

$$b(x, \xi) = \sum_{j=0}^{N} c_j(x, \xi)\xi_j, \tag{4.5}$$

where, for $0 \leq j \leq N$,

$$c_j \text{ is an equicontinuous } C^0 \text{ bundle map,} \tag{4.6}$$

$$c_j \text{ is of class } C^{0,s} \text{ semi-uniformly in the base variable,} \tag{4.7}$$

$$c_j(\cdot, 0) \in L^\infty(\mathbb{R}^N). \tag{4.8}$$

As in Section 2, we finally assume that

$$0 < \delta^\infty := \underline{\lim}_{|x|\to\infty}\, c_0(x,0). \tag{4.9}$$

Remark 4.1: Conditions (4.5) to (4.8) hold if b is a C^1_ξ bundle map with $b(\cdot,0)=0$ and $\partial_{\xi_j} b$ is of class $C^{0,s}$ locally uniformly with respect to the base variable with $\partial_{\xi_j} b(\cdot,0) \in L^\infty(\mathbb{R}^N), 0 \le j \le N$. This follows at once from Remark 2.1 and Theorem 3.2. $\square$

We shall need two preliminary lemmas which are standard results from linear elliptic PDE theory. Given an open subset $\Omega \subset \mathbb{R}^N$, we consider the linear second-order differential operator on Ω

$$L := -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{N} B_\alpha(x)\partial_\alpha + C(x). \tag{4.10}$$

Lemma 4.2. *([6, Theorem 9.19, p. 243]): Let $\Omega \subset \mathbb{R}^N$ be an open subset and let L in (4.10) be elliptic in Ω. Let $q \in (1,\infty)$ and $u \in W^{2,q}_{\mathrm{loc}}(\Omega)$. If for some $0 < \sigma < 1$ the coefficients of L are in $C^{0,\sigma}(\Omega)$ and $Lu \in C^{0,\sigma}(\Omega)$, then $u \in C^{2,\sigma}(\Omega)$.*

Lemma 4.3. *([6, Corollary 6.3, p. 93]): Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset and L in (4.10) be a strictly elliptic operator on Ω. Suppose that the coefficients of L are of class $C^{0,\sigma}(\overline{\Omega})$ for some $0 < \sigma < 1$. Let $u \in C^{2,\sigma}(\Omega)$ be such that $Lu \in C^{0,\sigma}(\overline{\Omega})$. Then, for every open subset $\Omega' \subset\subset \Omega$, there is a constant $M > 0$ such that $|u|_{1,\infty,\Omega'} + |u|_{2,\infty,\Omega'} \le M(|u|_{0,\infty,\Omega} + [[Lu]]_{0,\sigma,\overline{\Omega}})$, and M depends only upon the ellipticity constant of L on Ω, the $C^{0,\sigma}(\overline{\Omega})$-norms of the coefficients of L, the diameter of Ω and the distance dist$(\Omega', \partial\Omega)$.*

Remark 4.2: The constant M in Lemma 4.3 depends in fact upon *estimates* of the various quantities mentioned in Lemma 4.3, i.e. the exact values are not needed. Furthermore, M is increased while remaining finite by using upper estimates for the $C^{0,\sigma}(\overline{\Omega})$ norms of the coefficients and/or positive lower estimates for the ellipticity constant. $\square$

If $u \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $q \in (1,\infty)$ is a solution of (2.1), then $v = u$ can be viewed as a solution of the linear problem $Lv = f$ where L is as in (4.10) and

$$A_{\alpha\beta}(x) := a_{\alpha\beta}(x, u(x), \nabla u(x)), \quad 1 \leq \alpha, \beta \leq N, \tag{4.11}$$

$$B_{\alpha}(x) := c_{\alpha}(x, u(x), \nabla u(x)), \quad 1 \leq \alpha \leq N, \tag{4.12}$$

$$C(x) := c_0(x, u(x), \nabla u(x)). \tag{4.13}$$

Remark 4.3: The coefficients $A_{\alpha\beta}(x), B_{\alpha}(x)$ and $C(x)$ above are what we called $p_{\alpha\beta}(x), q_{\alpha}(x)$ and $q_0(x)$, respectively, in the proof of Theorem 2.1. The operator L, however, is not the same as in that proof. $\square$

Lemma 4.4. *Let $p \in (N, \infty)$ and let $u \in W^{2,p}(\mathbb{R}^N)$ be a solution of (2.1) with $f \in L^p(\mathbb{R}^N)$. Let $r > 0$ and $0 < \sigma \leq s(1 - N/p)$ be fixed, where $0 < s \leq 1$ is chosen as in (4.2) and (4.7). Let L denote the differential operator (4.10) with coefficients given by (4.11), (4.12) and (4.13). Then,*

(i) The coefficients of L are of class $C^{0,\sigma}(\mathbb{R}^N)$.

(ii) The $C^{0,\sigma}(\overline{B}_r)$-norms of the coefficients of L are uniformly bounded independently of the center of the ball $B_r \subset \mathbb{R}^N$ with radius $r > 0$.

(iii) There are constants $R > 0$ and $\lambda > 0$ such that $\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\eta_\alpha\eta_\beta \geq \lambda|\eta|^2$ for every $x \in \tilde{B}_R$ and every $\eta \in \mathbb{R}^N$.

(iv) If the coefficients $a_{\alpha\beta}$ and c_j depend only upon x and $\xi_0, 1 \leq \alpha, \beta \leq N, 0 \leq j \leq N$, (i) (ii) and (iii) above hold with $u \in W^{1,p}(\mathbb{R}^N)$.

Proof. Parts (i) and (ii) follow from the definitions (4.11) to (4.13) of the coefficients of L, conditions (4.1) - (4.2) and (4.6) - (4.7), and Theorem 3.1 (i). For the proof of (iii), recall that $\lim_{|x|\to\infty} |u(x)| + |\nabla u(x)| = 0$, hence $(u(x), \nabla u(x)) \in \overline{B}(0, \epsilon)$ where $\epsilon > 0$ is chosen as in condition (4.4′) provided that $|x|$ is large enough, say $|x| \geq R$ with R as in condition (4.4) (after increasing it if necessary). By Lemma 4.1, $\lambda = \gamma/2$ works.

That (iv) holds follows from obvious modifications of the above arguments: Use Theorem 3.1 (ii) instead of (i) and note that the validity of Lemma 4.1 requires only $|\xi_0| < \epsilon$ if the coefficients $a_{\alpha\beta}$ are independent of $(\xi_1, \cdots, \xi_N)$. $\square$

Theorem 4.1. *Retain assumptions (4.1) to (4.9) and let $\mu^* > 0$([2]) be defined by (2.14). Let $p \in (N, \infty), q \in (N, \infty)$ and let $f \in L^p(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ be such that $\overline{\lim}_{|x|\to\infty} e^{\nu|x|}|f(x)| < \infty$ and $\overline{\lim}_{a\to\infty} e^{\nu a}|f|_{1,q,\tilde{B}_a} < \infty$ (e.g. $\overline{\lim}_{|x|\to\infty} e^{\nu|x|}|\nabla f(x)| < \infty$) for some $\nu > 0$. Then, every solution $u \in W^{2,p}(\mathbb{R}^N)$ of (2.1) is of class C^2 in $\tilde{B}_R$ for $R > 0$ large enough (but depending upon u) and for any $\mu < \min(\mu^*, \nu)$ and every $\kappa \in \mathbb{N}^N$ with $|\kappa| \leq 2$ we have $\lim_{|x|\to\infty} e^{\mu|x|} D^\kappa u(x) = 0$.*

Proof. The case $\kappa = 0$ is covered by Corollary 2.1 (by (4.4′), condition (2.5′) holds, whence (2.10) is vacuous). We henceforth assume $|\kappa| = 1$ or 2. In what follows, $r > 0$ and $0 < \sigma \leq \min(s(1 - N/p), s(1 - N/q))$ are fixed once and for all, where $0 < s \leq 1$ is as in (4.2) and (4.7).

Rewrite (2.1) as the linear equation $Lv = f$ for $v = u$, where L is the operator (4.10) with coefficients given by (4.11), (4.12) and (4.13). From Lemma 4.4 (i), these coefficients are in $C^{0,\sigma}(\mathbb{R}^N)$, as is the right-hand side f since $0 < \sigma \leq 1 - N/q$ (so that $W^{1,q}(\mathbb{R}^N) \hookrightarrow C^{0,\sigma}(\mathbb{R}^N)$). Part (iii) of Lemma 4.4 also yields that L is (strictly) elliptic on $\tilde{B}_R$ for $R > 0$ large enough. It thus follows from Lemma 4.2 with $\Omega = \tilde{B}_R$ that $u \in C^{2,\sigma}(\tilde{B}_R)$.

In Lemma 4.3, choose $\Omega = B_r$, an open ball with arbitrary center $x_0 \in \tilde{B}_{R+r}$ and radius $r > 0$ (so that $B_r \subset \tilde{B}_R$), and (say) $\Omega' = B_{r/2}$, the open ball with same center x_0 and radius $r/2$. The corresponding inequality in Lemma 4.3 reads

$$|u|_{1,\infty,B_{r/2}} + |u|_{2,\infty,B_{r/2}} \leq M(|u|_{0,\infty,B_r} + [[f]]_{0,\sigma,\overline{B}_r}), \tag{4.14}$$

where, by Lemma 4.4 (ii) and (iii) (and Remark 4.2) the constant M is independent of the center x_0 of B_r. Since $|\kappa| = 1$ or 2 and x_0 is the center of $B_{r/2}$ as well, (4.14) implies

$$|D^\kappa u(x_0)| \leq M(|u|_{0,\infty,B_r} + [[f]]_{0,\sigma,\overline{B}_r}), \tag{4.15}$$

with no modification of M.

The embedding constant $W^{1,q}(B_r) \hookrightarrow C^{0,\sigma}(\overline{B}_r)$ depends upon q, σ and r, but is independent of x_0 (an argument already used in the proof of Theorem 3.1). Accordingly, after

([2])In this section, $\mu^* = \infty$ is ruled out by condition (4.4).

modifying M in (4.15) in a way independent of x_0, we obtain

$$|D^\kappa u(x_0)| \leq M(|u|_{0,\infty,B_r} + ||f||_{1,q,B_r}). \tag{4.16}$$

¿From Corollary 2.1, we may assume that $R > 0$ above is also large enough that

$$|u(x)| \leq e^{-\mu|x|} \text{ for } |x| \geq R. \tag{4.17}$$

Since $\overline{\lim}_{|x|\to\infty} e^{\nu|x|}|f(x)| < \infty$ and $\overline{\lim}_{a\to\infty} e^{\nu a}|f|_{1,a,\tilde{B}_a} < \infty$, and since $\mu < \nu$, we have $\lim_{a\to\infty} e^{\mu a}||f||_{1,q,\tilde{B}_a} = 0$, so that for $a > 0$ large enough

$$||f||_{1,q,\tilde{B}_a} \leq e^{-\mu a}. \tag{4.18}$$

Let now $R > 0$ be fixed. Since $x_0 \in \tilde{B}_{R+r}$, we have

$$B_r \subset \tilde{B}_{|x_0|-r} \subset \tilde{B}_R. \tag{4.19}$$

¿From (4.17) and (4.19), we infer that

$$|u|_{0,\infty,B_r} \leq e^{\mu r} e^{-\mu|x_0|}, \tag{4.20}$$

while (4.18) with $a = |x_0| - r$ and (4.19) provide

$$||f||_{1,q,B_r} \leq ||f||_{1,q,\tilde{B}_{|x_0|-r}} \leq e^{\mu r} e^{-\mu|x_0|}. \tag{4.21}$$

By substitution of (4.20) and (4.21) into (4.16), we find $|D^\kappa u(x_0)| \leq M e^{\mu r} e^{-\mu|x_0|}$. This holds for every $|x_0| > R+r$. Thus, $\overline{\lim}_{|x|\to\infty} e^{\mu|x|}|D^\kappa u(x)| < \infty$. By replacing μ by μ' with $\mu < \mu' < \min(\mu^*, \nu)$, we get $\lim_{|x|\to\infty} e^{\mu|x|} D^\kappa u(x) = 0$, and the proof is complete. □

Theorem 4.2. *Suppose that the coefficients $a_{\alpha\beta}$ and $c_j, 1 \leq \alpha, \beta \leq N, 0 \leq j \leq N$, depend only upon x and ξ_0. Then, Theorem 4.1 remains valid for $p \in (N, \infty)$ and $u \in W^{1,p}(\mathbb{R}^N) \cap W^{2,N}_{\text{loc}}(\mathbb{R}^N)$.*

Proof. In the proof of Theorem 4.1, use Theorem 2.2 instead of 2.1, and Lemma 4.4 (iv). Note that Lemma 4.2 can still be used to get $u \in C^{2,\sigma}(\mathbb{R}^N)$ since $u \in W^{2,N}_{\text{loc}}(\mathbb{R}^N)$. $\square$

The following corollary complements Theorem 4.1 when $p \in (N/2, N], p > 1$ and the coefficients depend only upon x and ξ_0.

Corollary 4.1. *Suppose that the coefficients $a_{\alpha\beta}$ and $c_j, 1 \leq \alpha, \beta \leq N, 0 \leq j \leq N$, depend only upon x and ξ_0. Then, Theorem 4.1 remains valid for the solutions $u \in W^{2,p}(\mathbb{R}^N)$, $p \in (N/2, \infty)$, $p > 1$, provided that, in addition, $q \leq p^* := Np/(N-p)$ when $p \in (N/2), N)$.*

Proof. There is nothing to prove if $p \in (N, \infty)$. If $p \in (N/2, N]$, use the embedding $W^{2,p}(\mathbb{R}^N) \hookrightarrow W^{1,q}(\mathbb{R}^N)$ and Theorem 4.2 with p replaced by q. $\square$

Corollary 4.2. *(i) Under the assumptions of Theorem 4.1 (resp. Corollary 4.1) every solution $u \in W^{2,p}(\mathbb{R}^N)$ of (2.1) with $p \in (N, \infty)$ (resp. $p \in (N/2, \infty), p > 1$) is in $W^{2,q}(\tilde{B}_R) \cap C^2(\tilde{B}_R)$ for $R > 0$ large enough (but depending upon u) and every $1 \leq q \leq \infty$. In particular, $u \in W^{2,q}(\mathbb{R}^N)$ for $1 \leq q \leq p$.*

(ii) If, in addition to condition (4.4), there is a constant $\gamma(x, \xi) > 0$ such that $\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, \xi)$ $\eta_\alpha \eta_\beta \geq \gamma(x, \xi)|\eta|^2$ for every $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N+1}$ and every $\eta \in \mathbb{R}^N$, then $u \in W^{2,q}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ for every $1 \leq q \leq \infty$.

Proof. (i) From either Theorem 4.1 or Corollary 4.1, u is of class C^2 in $\overline{\tilde{B}_R}$ for $R > 0$ large enough and the derivatives $D^\kappa u$ have exponential decay for $|\kappa| \leq 2$. This implies at once $u \in W^{2,q}(\tilde{B}_R)$ for $1 \leq q \leq \infty$. For $1 \leq q \leq p$, we also have $W^{2,p}(\mathbb{R}^N) \hookrightarrow W^{2,q}_{\text{loc}}(\mathbb{R}^N)$, whence $u \in W^{2,q}(\mathbb{R}^N)$.

(ii) Under the extra ellipticity condition, the operator L in (4.10) with coefficients given by (4.11), (4.12) and (4.13) is elliptic on $\mathbb{R}^N$. By Lemma 4.4, its coefficients are of class

$C^{0,\sigma}(\mathbb{R}^N)$ for some $0 < \sigma \leq 1$ and $Lu = f \in C^{0,\sigma}(\mathbb{R}^N)$. Thus, $u \in C^{2,\sigma}(\mathbb{R}^N)$ by Lemma 4.2. Hence $u \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for $1 \leq q \leq \infty$ and, from part (i), $u \in W^{2,q}(\mathbb{R}^N)$. $\square$

Corollary 4.3. *Let* $L := -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\partial_\alpha\partial_\beta + \sum_{\alpha=1}^{N} B_\alpha(x)\partial_\alpha + C(x)$ *be a strictly elliptic linear operator on* $\mathbb{R}^N$ *with bounded coefficients semi-uniformly of class* $C^{0,s}$(3) *for some* $0 < s \leq 1$. *Suppose that* $A_{\alpha\beta} = A_{\beta\alpha}, 1 \leq \alpha, \beta \leq N$ *and that* $\delta^\infty := \underline{\lim}_{|x|\to\infty} C(x) > 0$. *Then, Theorem 4.1, Corollary 4.1 and both parts of Corollary 4.2 are valid for the solutions of the equation* $Lu = f$ *if* f *satisfies the conditions required in those results. In particular, if* $p \in (N/2, \infty), p > 1, \mu < \mu^*$ *and* u *is an element of the generalized null-space of* L *in* $W^{2,p}(\mathbb{R}^N)$, *then* $\lim_{|x|\to\infty} e^{\mu|x|} D^\kappa u(x) = 0$ *for* $|\kappa| \leq 2$.

Remark 4.4: The results of this section are not as easily extended to arbitrary unbounded domains as those of Section 2 (see Remark 2.4). However, the procedure continues to work in exterior domains Ω since the following two basic features are preserved: (i) Every solution u also solves a linear strictly elliptic equation at large distances and (ii) There is $r > 0$ such that the open ball $B(x_0, r)$ is contained in Ω for all $x_0 \in \Omega$ with $|x_0|$ large enough. It is obvious that (ii) breaks down whenever $\partial\Omega$ is unbounded. Without going into technicalities, this issue can be resolved as follows. First, the exponential decay of the derivatives of the solutions u with Supp $u_{|\partial\Omega}$ compact holds in the open subset $\Omega_\epsilon := \{x \in \Omega : \text{ dist } (x, \partial\Omega) > \epsilon\}$ for every $\epsilon > 0$. Of course, this is meaningful only when Ω_ϵ is unbounded for $\epsilon > 0$ small enough, i.e. Ω does not "thin out" at infinity. Assuming this, the same conclusion in Ω, not only Ω_ϵ, can be obtained by involving the Hölder estimates near the boundary, under suitable geometric assumptions about $\partial\Omega$ including (but not limited to) Lipschitz-continuity. $\square$

5. Pohozaev identities.

We now turn to identities of Pohozaev type for the $W^{2,p}$ solutions of second-order quasilinear equations on $\mathbb{R}^N$ in a divergence form satisfying mild ellipticity conditions. For simplicity, we confine our attention to the case when the coefficients are x-independent. The general case is discussed in Remark 5.2.

(3)See Definition 3.1.

We shall consider equations of the form

$$(5.1) \qquad -\sum_{\alpha=1}^{N}\partial_\alpha[A_\alpha(u,\nabla u)]+A_0(u,\nabla u)+\varphi(u)=0$$

where

$$(5.2) \qquad A_j=\partial_{\xi_j}Q, \quad 0\le j\le N,$$

for some function $Q:\mathbb{R}^{N+1}\to\mathbb{R}$ of class $C^{2,s}$ for some $0<s\le 1$ such that

$$(5.3) \qquad \sum_{\alpha,\beta=1}^{N}\partial_{\xi_\alpha}\partial_{\xi_\beta}Q(0)\eta_\alpha\eta_\beta\ge\gamma|\eta|^2, \quad \forall\,\eta\in\mathbb{R}^N,$$

where $\gamma>0$ is a constant. A useful normalization of the problem consists in replacing $Q(\xi)$ by $Q(\xi)-Q(\xi_0,0)-\sum_{\alpha=1}^{N}A_\alpha(0)\xi_\alpha$ and $\varphi(\xi_0)$ by $\varphi(\xi_0)+A_0(\xi_0,0)$. This does not change (5.1), (5.2) or (5.3) provided that $A_\alpha(\xi)$ is changed into $A_\alpha(\xi)-A_\alpha(0)$ and $A_0(\xi)$ is changed into $A_0(\xi)-A_0(\xi_0,0)$ and introduces the simplifications

$$(5.4) \qquad A_0(\xi_0,0)\ (=\partial_{\xi_0}Q(\xi_0,0))=0, \quad \forall\,\xi_0\in\mathbb{R},$$

$$(5.5a) \qquad Q(\xi_0,0)=0,$$

$$(5.5b) \qquad A_\alpha(0)\ (=\partial_{\xi_\alpha}Q(0))=0, \quad 1\le\alpha\le N.$$

The function $\varphi:\mathbb{R}\to\mathbb{R}$ will be of class $C^{1,s}$ with $0<s\le 1$ as above and satisfy

$$(5.6) \qquad \varphi(0)=0,$$

$$(5.7) \qquad \delta^\infty:=\varphi'(0)>0.$$

The formal use of the chain rule in (5.1) shows that u solving (5.1) must also solve a problem of the form (2.1). The following lemma, which elaborates upon [6, Lemma 7.5, p. 151], will allow us to justify this procedure:

Lemma 5.1. *(i) Let $g : \mathbb{R}^{N+1} \to \mathbb{R}$ be of class C^1 with $g(0) = 0$. For $p \in (N, \infty)$, the mapping $u \in W^{2,p}(\mathbb{R}^N) \longmapsto g(u, \nabla u) \in W^{1,p}(\mathbb{R}^N)$ is well defined and the chain rule holds:*

$$\partial_\alpha[g(u, \nabla u)] = \partial_{\xi_0} g(u, \nabla u)\partial_\alpha u + \sum_{\beta=1}^{N} \partial_{\xi_\beta} g(u, \nabla u)\partial_\alpha \partial_\beta u, \ 1 \le \alpha \le N. \tag{5.8}$$

(ii) Let $g : \mathbb{R} \to \mathbb{R}$ be of class C^1 with $g(0) = 0$. For $p \in (N, \infty)$, the mapping $u \in W^{1,p}(\mathbb{R}^N) \longmapsto g(u) \in L^p(\mathbb{R}^N)$ is well defined.

Note: In (ii), it is even true that $g(u) \in W^{1,p}(\mathbb{R}^N)$ and that the chain rule is valid, but this will not be needed here.

Proof. (i) Denote by $\mathbf{g}$ the operator $u \longmapsto g(u, \nabla u)$. It is shown in [17, Theorem 2.3] that $\mathbf{g}$ is continuous from $W^{2,p}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$. Also, from [17, Theorem 2.1], the operators $\gamma_j : u \longmapsto (\partial_{\xi_j} g)(u, \nabla u)$ are continuous from $W^{2,p}(\mathbb{R}^N)$ to $L^\infty(\mathbb{R}^N), 0 \le j \le N$. It follows that the right-hand side of (5.8), i.e. $\mathbf{\Gamma}_\alpha(u) := \gamma_0(u)\partial_\alpha u + \sum_{\beta=1}^{N} \gamma_\beta(u)\partial_\alpha\partial_\beta u$, is continuous from $W^{2,p}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$.

Relation (5.8), i.e. $\partial_\alpha[\mathbf{g}(u)] = \mathbf{\Gamma}_\alpha(u)$, holds when $u \in \mathcal{D}(\mathbb{R}^N)$ since g is C^1. If $u \in W^{2,p}(\mathbb{R}^N)$ is the limit of the sequence (u_n) from $\mathcal{D}(\mathbb{R}^N)$, the continuity of $\mathbf{\Gamma}_\alpha$ implies that $\mathbf{\Gamma}_\alpha(u_n)$ tends to $\mathbf{\Gamma}_\alpha(u)$ in $L^p(\mathbb{R}^N)$. Since $\mathbf{\Gamma}_\alpha(u_n) = \partial_\alpha[\mathbf{g}(u_n)]$, the sequence $(\partial_\alpha[\mathbf{g}(u_n)])$ is Cauchy in $L^p(\mathbb{R}^N)$, $1 \le \alpha \le N$. By the continuity of $\mathbf{g}$, the same thing is true of the sequence $(\mathbf{g}(u_n))$. Thus, $(\mathbf{g}(u_n))$ is Cauchy in $W^{1,p}(\mathbb{R}^N)$, and its limit must be its limit $\mathbf{g}(u)$ in $L^p(\mathbb{R}^N)$. This shows that $\mathbf{g}(u) \in W^{1,p}(\mathbb{R}^N)$ and that $\mathbf{g}(u_n) \to \mathbf{g}(u)$ in $W^{1,p}(\mathbb{R}^N)$.

In turn, $\mathbf{g}(u_n) \to \mathbf{g}(u)$ in $W^{1,p}(\mathbb{R}^N)$ implies that $\partial_\alpha[\mathbf{g}(u_n)](= \mathbf{\Gamma}_\alpha(u_n))$ tends to $\partial_\alpha[\mathbf{g}(u)]$ in $L^p(\mathbb{R}^N), 1 \le \alpha \le N$. Since $\mathbf{\Gamma}_\alpha(u_n) \to \mathbf{\Gamma}_\alpha(u)$ in $L^p(\mathbb{R}^N)$, we infer that $\partial_\alpha[\mathbf{g}(u)] = \mathbf{\Gamma}_\alpha(u)$, i.e. (5.8) holds.

(ii) Write $g(u) = h(u)u$ with $h(\xi_0) = g(\xi_0)/\xi_0$ if $\xi_0 \neq 0$ and $h(0) = g'(0)$. Then, h is continuous, so that $h(u)$ is continuous and bounded since $u \in W^{1,p}(\mathbb{R}^N) \hookrightarrow C^0_d(\mathbb{R}^N)$. Hence, $g(u) = h(u)u \in L^p(\mathbb{R}^N)$. $\square$

¿From Lemma 5.1 (i) with $g(\xi) = A_\alpha(\xi), 1 \le \alpha \le N$, (see (5.5b)), we get

Lemma 5.2. *Let $p \in (N, \infty)$. Then, $u \in W^{2,p}(\mathbb{R}^N)$ solves equation (5.1) if and only if*

$$(5.9) \qquad -\sum_{\alpha,\beta=1}^{N}(\partial_{\xi_\beta}A_\alpha)(u,\nabla u)\partial_\alpha\partial_\beta u + A_0(u,\nabla u) - \sum_{\alpha=1}^{N}(\partial_{\xi_0}A_\alpha)(u,\nabla u)\partial_\alpha u + \varphi(u) = 0.$$

We shall set

$$\rho^\infty := \text{ spectral radius of } (\partial_{\xi_\alpha}\partial_{\xi_\beta}Q(0)),$$

so that $\rho^\infty > 0$ by (5.3).

Theorem 5.1. *Retain assumptions (5.2) to (5.7) and set $\mu^* := \sqrt{\rho^\infty/\delta^\infty}$. Let $p \in (N, \infty)$ and let $u \in W^{2,p}(\mathbb{R}^N)$ be a solution of (5.1). Then, $u \in C^2(\tilde{B}_R)$ for $R > 0$ large enough and for every $\mu < \mu^*$, every $\kappa \in \mathbb{N}^N$ with $|\kappa| \le 2$ and every polynomial map $P = P(x)$, we have $\lim_{|x|\to\infty} e^{\mu|x|}D^\kappa u(x) = 0$ and $PD^\kappa u \in W^{2-|\kappa|,q}(\tilde{B}_R)$ for every $1 \le q \le \infty$.*

Proof. From Lemma 5.2, every solution $u \in W^{2,p}(\mathbb{R}^N)$ of (5.1) solves (5.9). Now, (5.9) is an equation of the form (2.1) with $a_{\alpha\beta}(\xi) = \partial_{\xi_\beta}A_\alpha(\xi) = \partial_{\xi_\beta}\partial_{\xi_\alpha}Q(\xi) = \partial_{\xi_\alpha}\partial_{\xi_\beta}Q(\xi)$ and with $b(\xi) = A_0(\xi) - \sum_{\alpha=1}^{N}\partial_{\xi_0}A_\alpha(\xi)\xi_\alpha + \varphi(\xi_0)$. It is straightforward to check that these (x-independent) coefficients $a_{\alpha\beta}$ and b satisfy all the conditions required in Section 4 for the validity of Theorem 4.1. We only point out that (4.5) holds, along with (4.6) and (4.7), because $b = b_0 + b_1$ with $b_0(\xi) := A_0(\xi) + \varphi(\xi_0)$ and $b_1(\xi) := -\sum_{\alpha=1}^{N}\partial_{\xi_0}A_\alpha(\xi)\xi_\alpha$. Since b_0 is of class $C^{1,s}$, Remark 4.1 ensures that $b_0(\xi) = \sum_{j=0}^{N}c_{0j}(\xi)\xi_j$ with $c_{0j} \in C^{0,s}(\mathbb{R}^{N+1}), 0 \le j \le N$, and $b_1(\xi)$ is given in the form $b_1(\xi) = \sum_{j=0}^{N}c_{1j}(\xi)\xi_j$ with $c_{10} = 0$ and $c_{1,\alpha} = -\partial_{\xi_0}A_\alpha = -\partial_{\xi_0}\partial_{\xi_\alpha}Q \in C^{0,s}(\mathbb{R}^N), 1 \le \alpha \le N$.

Also, δ^∞ and ρ^∞ in (5.7) and (5.10), respectively, coincide with the corresponding values used in Theorem 4.1 (see (2.11) and (2.12)) since $c_\alpha(\xi) = c_{0\alpha}(\xi) + c_{1\alpha}(\xi) = \int_0^1 \partial_{\xi_\alpha}\partial_{\xi_0}Q(t\xi)dt - \partial_{\xi_0}\partial_{\xi_\alpha}Q(\xi)$ for $1 \le \alpha \le N$, we have $c_\alpha(0) = 0$ and hence $c^\infty = 0$ in (2.13). The conclusion follows from Theorem 4.1 and Corollary 4.2 with $f = 0$. $\square$

Much of the technicalities associated with the proof of Pohozaev's identities (Theorem 5.2) are handled by a simple corollary of Theorem 5.1. Before stating it, we introduce

$$(5.11) \qquad \Phi(\xi_0) := \int_0^{\xi_0}\varphi(t)dt$$

so that Φ is of class C^2 (even $C^{2,s}$) and $\Phi(0) = 0$. For convenient reference, we also state an elementary lemma, whose proof follows from the denseness of $\mathcal{D}(\mathbb{R}^N)$ in $W^{1,1}(\mathbb{R}^N)$.

Lemma 5.3. *For $v \in W^{1,1}(\mathbb{R}^N)$ and $1 \leq \alpha \leq N$, we have $\int_{\mathbb{R}^N} \partial_\alpha v = 0$.*

Corollary 5.1. *Retain assumptions (5.2) to (5.7). Let $p \in (N, \infty)$ and let $u \in W^{2,p}(\mathbb{R}^N)$ be a solution of (5.1). Then, the following properties hold:*

(i) The real vector space $\mathcal{E}_u$ generated by $u, \partial_\alpha u, 1 \leq \alpha \leq N, \Phi(u), \varphi(u), Q(u, \nabla u)$ and $A_j(u, \nabla u), 0 \leq j \leq N$, is a subspace of $W^{1,p}(\mathbb{R}^N) \cap C^1(\tilde{B}_R)$ for $R > 0$ large enough.

(ii) The space $\mathcal{E}_u$ in (i) generates a subalgebra $\mathcal{A}_u$ of $W^{1,p}(\mathbb{R}^N) \cap C^1(\tilde{B}_R)$.

(iii) If $v \in \mathcal{A}_u$, then $D^\kappa v \in L^p(\mathbb{R}^N) \cap C^0(\tilde{B}_R)$ has exponential decay for every $\kappa \in \mathbb{N}^N$ with $|\kappa| \leq 1$. As a result, the real vector space $\tilde{\mathcal{E}}_u$ generated by the elements $D^\kappa v, v \in \mathcal{A}_u, |\kappa| \leq 1$, is a subspace of $L^p(\mathbb{R}^N) \cap C^0(\tilde{B}_R)$ whose elements have exponential decay.

(iv) The $\mathcal{A}_u$-module $\tilde{\mathcal{M}}_u$ generated by $\tilde{\mathcal{E}}_u$ is contained in $L^p(\mathbb{R}^N) \cap C^0(\tilde{B}_R)$. Furthermore, for every polynomial map $P = P(x)$ and every $w \in \tilde{\mathcal{M}}_u$, we have $Pw \in L^1(\mathbb{R}^N)$.

(v) If $v \in \mathcal{A}_u$ and $P(x)$ is a polynomial map, then $Pv \in W^{1,1}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \partial_\alpha(Pv) = 0$ for $1 \leq \alpha \leq N$.

Proof. (i) That all the generators of $\mathcal{E}_u$ are in $W^{1,p}(\mathbb{R}^N)$ follows from Lemma 5.1 (i) and conditions (5.4), (5.5) and (5.6) (and $\Phi(0) = 0$). Also, $u \in C^2(\tilde{B}_R)$ for R large enough by Theorem 5.1, whence all the generators of $\mathcal{E}_u$ are in $C^1(\tilde{B}_R)$.

(ii) This follows from (i) and from $W^{1,p}(\mathbb{R}^N)$ being an algebra for $p > N$.

(iii) Since $\varphi(0) = \Phi(0) = 0$ and $Q(0) = A_j(0) = 0, 0 \leq j \leq N$, the mean-value theorem and the exponential decay of u and ∇u (Theorem 5.1) show that the generators of $\mathcal{E}_u$ (hence of $\mathcal{A}_u$) have exponential decay at infinity. By the chain rule (5.8) and $u \in C^1_d(\mathbb{R}^N)$, it is readily checked that the exponential decay of the second derivatives of u (Theorem 5.1) implies that the gradients of the generators of $\mathcal{A}_u$ have exponential decay. By an easy induction argument, it follows that any finite product of generators and its gradient have exponential decay (the product rule is valid since $\mathcal{A}_u$ is a subalgebra of $W^{1,p}(\mathbb{R}^N) \cap C^1(\tilde{B}_R)$). Thus, if $v \in \mathcal{A}_u$, both v and ∇v have exponential decay. That the elements of $\tilde{\mathcal{E}}_u$ have exponential decay is then obvious.

(iv) If $v \in \mathcal{A}_u$, then $v \in W^{1,p}(\mathbb{R}^N) \subset C^0_d(\mathbb{R}^N)$ by (ii) and hence v is bounded on $\mathbb{R}^N$. Since $g \in \tilde{\mathcal{E}}_u$ implies $g \in L^p(\mathbb{R}^N) \cap C^0(\tilde{B}_R)$ by (iii), we have $vg \in L^p(\mathbb{R}^N) \cap C^0(\tilde{B}_R)$. Thus, $\tilde{\mathcal{M}}_u \subset L^p(\mathbb{R}^N) \cap C^0(\tilde{B}_R)$.

By (iii) $g \in \tilde{\mathcal{E}}_u$ and $v \in \mathcal{A}_u$ have exponential decay, whence Pvg has exponential decay. In particular, $Pvg \in L^1(\tilde{B}_R)$ for $R > 0$ large enough. Also, $Pvg \in L^p_{\text{loc}}(\mathbb{R}^N)$ since $vg \in L^p(\mathbb{R}^N)$ from the above. From the embedding $L^p_{\text{loc}}(\mathbb{R}^N) \hookrightarrow L^1_{\text{loc}}(\mathbb{R}^N)$, we infer that $Pvg \in L^1(\mathbb{R}^N)$. This suffices to prove (iv).

(v) If $v \in \mathcal{A}_u$, then v and $\partial_\beta v, 1 \le \beta \le N$, are in $\tilde{\mathcal{E}}_u \subset \tilde{\mathcal{M}}_u$. Thus, $Pv, P\partial_\beta v$ and $(\partial_\beta P)v$ are in $L^1(\mathbb{R}^N)$ by (iv). By the product rule (valid here because P is a polynomial, hence a C^∞ map) this is the same as saying that $Pv \in W^{1,1}(\mathbb{R}^N)$. That $\int_{\mathbb{R}^N} \partial_\alpha(Pv) = 0, 1 \le \alpha \le N$, follows from Lemma 5.3. $\square$

Theorem 5.2. *(Pohozaev identities): Retain assumptions (5.2) to (5.7). Let $p \in (N, \infty)$ and let $u \in W^{2,p}(\mathbb{R}^N)$ be a solution of (5.1). We have*

(i) The functions $A_0(u, \nabla u)u, A_\alpha(u, \nabla u)\partial_\alpha u, 1 \le \alpha \le N, Q(u, \nabla u), \varphi(u)u$ and $\Phi(u)$ are all in $L^1(\mathbb{R}^N)$.

(ii) $\int_{\mathbb{R}^N} A_0(u, \nabla u)u + \int_{\mathbb{R}^N} \varphi(u)u = -\sum_{\alpha=1}^{N} \int_{\mathbb{R}^N} A_\alpha(u, \nabla u)\partial_\alpha u = -N\{\int_{\mathbb{R}^N} \Phi(u) + \int_{\mathbb{R}^N} Q(u, \nabla u)\}.$

Proof. In this proof, $\mathcal{A}_u, \tilde{\mathcal{E}}_u$ and $\tilde{\mathcal{M}}_u$ refer to the notation used in Corollary 5.1.

(i) All these functions are from $\mathcal{A}_u \subset W^{1,1}(\mathbb{R}^N)$ (Corollary 5.1 (v)).

(ii) Given $1 \le \alpha \le N$, we have $A_\alpha(u, \nabla u) \in \mathcal{A}_u$ and $u \in \mathcal{A}_u$. Since $\mathcal{A}_u$ is a subalgebra of $W^{1,p}(\mathbb{R}^N)$ (Corollary 5.1 (ii)), the product rule holds, yielding $\partial_\alpha[A_\alpha(u, \nabla u)]u = \partial_\alpha[A_\alpha(u, \nabla u)u] - A_\alpha(u, \nabla u)\partial_\alpha u$. The right-hand side is in $\tilde{\mathcal{E}}_u \subset \tilde{\mathcal{M}}_u \subset L^1(\mathbb{R}^N)$ (Corollary 5.1 (iv)). Since also $\int_{\mathbb{R}^N} \partial_\alpha[A_\alpha(u, \nabla u)u] = 0$ by Corollary 5.1 (v), it follows that $\int_{\mathbb{R}^N} \partial_\alpha[A_\alpha(u, \nabla u)]u = -\int_{\mathbb{R}^N} A_\alpha(u, \nabla u)\partial_\alpha u$. Thus, multiplying (5.1) by u and integrating (and using (i)) we obtain the first identity in (ii) of the theorem.

The proof of the second identity is similar, but more involved: Since $\nabla u \in W^{1,p}(\mathbb{R}^N)$ and $D^\kappa u$ decays exponentially for $|\kappa| \le 2$ (Theorem 5.1), we have $x \cdot \nabla u \in W^{1,p}(\mathbb{R}^N)$. Hence, the product rule may be applied to $A_\alpha(u, \nabla u)(x \cdot \nabla u)$, yielding $\partial_\alpha[A_\alpha(u, \nabla u)](x \cdot \nabla u) = \partial_\alpha[A_\alpha(u, \nabla u)(x \cdot \nabla u)] - A_\alpha(u, \nabla u)\partial_\alpha(x \cdot \nabla u)$. Since $x_1, \cdots, x_N$ are polynomials, the product rule also applies to $x \cdot \nabla u$. This gives

$$\partial_\alpha[A_\alpha(u,\nabla u)](x\cdot\nabla u) = \partial_\alpha[A_\alpha(u,\nabla u)(x\cdot\nabla u)] - A_\alpha(u,\nabla u)\partial_\alpha u - \sum_{\beta=1}^{N} x_\beta A_\alpha(u,\nabla u)\partial_\alpha\partial_\beta u. \tag{5.12}$$

By Corollary 5.1 (v), we have $A_\alpha(u,\nabla u)(x\cdot\nabla u) \in W^{1,1}(\mathbb{R}^N)$, whence $\partial_\alpha[A_\alpha(u,\nabla u)(x\cdot\nabla u)] \in L^1(\mathbb{R}^N)$. Furthermore, Corollary 5.1 (v) also shows that

$$\int_{\mathbb{R}^N} \partial_\alpha[A_\alpha(u,\nabla u)(x\cdot\nabla u)] = 0. \tag{5.13}$$

The other terms in the right-hand side of (5.12) are in $L^1(\mathbb{R}^N)$ by Corollary 5.1 (iv). Thus, $\partial_\alpha[A_\alpha(u,\nabla u)](x\cdot\nabla u) \in L^1(\mathbb{R}^N)$. Upon integrating and using (5.13), we find

$$\int_{\mathbb{R}^N} \partial_\alpha[A_\alpha(u,\nabla u)](x\cdot\nabla u) = -\int_{\mathbb{R}^N} A_\alpha(u,\nabla u)\partial_\alpha u - \sum_{\beta=1}^{N} x_\beta A_\alpha(u,\nabla u)\partial_\alpha\partial_\beta u.$$

Since $A_0(u,\nabla u)(x\cdot\nabla u) \in L^1(\mathbb{R}^N)$ (Corollary 5.1 (iv)), this yields

$$-\sum_{\alpha=1}^{N}\int_{\mathbb{R}^N} \partial_\alpha[A_\alpha(u,\nabla u)](x\cdot\nabla u) + \int_{\mathbb{R}^N} A_0(u,\nabla u)(x\cdot\nabla u) =$$
$$\sum_{\alpha=1}^{N}\int_{\mathbb{R}^N} A_\alpha(u,\nabla u)\partial_\alpha u + \sum_{\beta=1}^{N}\int_{\mathbb{R}^N} x_\beta[A_0(u,\nabla u)\partial_\beta u + \sum_{\alpha=1}^{N} A_\alpha(u,\nabla u)\partial_{\alpha\beta} u].$$

Recalling that $A_j = \partial_{\xi_j} Q, 0 \le j \le N$, and since the chain rule (5.8) is valid for the evaluation of $\partial_\beta[Q(u,\nabla u)]$, this may be rewritten as

$$-\sum_{\alpha=1}^{N}\int_{\mathbb{R}^N} \partial_\alpha[A_\alpha(u,\nabla u)](x\cdot\nabla u) + \int_{\mathbb{R}^N} A_0(u,\nabla u)(x\cdot\nabla u) = \sum_{\alpha=1}^{N}\int_{\mathbb{R}^N} A_\alpha(u,\nabla u)\partial_\alpha u + \sum_{\beta=1}^{N}\int_{\mathbb{R}^N} x_\beta\partial_\beta[Q(u,\nabla u)]. \tag{5.14}$$

By the product rule, $x_\beta\partial_\beta[Q(u,\nabla u)] = \partial_\beta[x_\beta Q(u,\nabla u)] - Q(u,\nabla u)$. By Corollary 5.1 (iv), the right-hand side is in $L^1(\mathbb{R}^N)$, and also the integral of the first term is 0. Thus, $\int_{\mathbb{R}^N} x_\beta\partial_\beta[Q(u,\nabla u)] = -\int_{\mathbb{R}^N} Q(u,\nabla u)$. Making this substitution into (5.14), we obtain

$$-\sum_{\alpha=1}^{N}\int_{\mathbb{R}^N} \partial_\alpha[A_\alpha(u,\nabla u)](x\cdot\nabla u) + \int_{\mathbb{R}^N} A_0(u,\nabla u)(x\cdot\nabla u) = \sum_{\alpha=1}^{N}\int_{\mathbb{R}^N} A_\alpha(u,\nabla u)\partial_\alpha u - N\int_{\mathbb{R}^N} Q(u,\nabla u). \tag{5.15}$$

By similar, but much shorter, manipulations,

$$\int_{\mathbb{R}^N} \varphi(u)(x \cdot \nabla u) = -N \int_{\mathbb{R}^N} \Phi(u), \tag{5.16}$$

where $\Phi' = \varphi$ was used. To complete the proof, just notice that by adding (5.15) and (5.16) and using (5.1), we arrive at $\sum_{\alpha=1}^{N} \int_{\mathbb{R}^N} A_\alpha(u, \nabla u)\partial_\alpha u - N \int_{\mathbb{R}^N} Q(u, \nabla u) - N \int_{\mathbb{R}^N} \Phi(u) = 0$, which is the second identity in part (ii) of the theorem. □

We complete this section with an examination of the case when the coefficients A_0 and $\partial_{\xi_j} A_\alpha, 0 \leq j \leq N$, depend only upon ξ_0. The motivation for doing this is of course that every solution of (5.1) solves (5.9) (at least when $p \in (N, \infty)$), and the results of Section 4 suggest that Theorem 5.2 could still be valid when $p \in (N/2, \infty), p > 1$, in this case.

A useful remark is that since $A_j = \partial_{\xi_j} Q, 0 \leq j \leq N$, the above special case arises only when Q is quadratic in $(\xi_1, \cdots, \xi_N)$, with first and second degree coefficients also independent of ξ_0. By the "normalization" conditions (5.4) and (5.5), the zeroth-order term must vanish and hence $Q(\xi)$ can only have the form $Q(\xi) = \sum_{\alpha,\beta=1}^{N} d_{\alpha\beta}\xi_\alpha\xi_\beta$ with $d_{\alpha\beta} = d_{\beta\alpha} \in \mathbb{R}$. Furthermore, the ellipticity condition (5.3) requires the matrix $(d_{\alpha\beta})$ to be positive definite. Therefore, after a linear change of variable in $\mathbb{R}^N$, $Q(\xi)$ reduces to $Q(\xi) = |\xi'|^2/2, \xi' := (\xi_1, \cdots, \xi_N)$ and the equation (5.1) takes the familiar form

$$-\Delta u + \varphi(u) = 0, \tag{5.17}$$

with φ of class $C^{1,s}, \varphi(0) = 0$ and $\varphi'(0) > 0$.

Theorem 5.3. *Let $p \in (N/2, \infty), p > 1$, and let $u \in W^{2,p}(\mathbb{R}^N)$ be a solution of (5.17) with $\varphi(0) = 0, \varphi'(0) > 0$. Then, $|\nabla u| \in L^2(\mathbb{R}^N), \varphi(u)u, \Phi(u) \in L^1(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} \varphi(u)u = \left(\frac{N}{2} - 1\right) \int_{\mathbb{R}^N} |\nabla u|^2 + N \int_{\mathbb{R}^N} \Phi(u) = 0.$$

Proof. From the choice of p, there is always $q \in (N, \infty)$ such that $u \in W^{1,q}(\mathbb{R}^N)$. By Lemma 5.1 (ii), we have $\varphi(u) \in L^q(\mathbb{R}^N)$, hence $\Delta u \in L^q(\mathbb{R}^N)$. The standard elliptic regularity of the Laplacian ensures that $u \in W^{2,q}(\mathbb{R}^N)$. Since $q \in (N, \infty)$, Theorem 5.2 yields the desired result. $\square$

Remark 5.2: There is no conceptual difficulty in extending the results of this section to the case when both Q and φ also depend upon x. However, the list of technical assumptions increases substantially. A generalization of Lemma 5.1 is needed when $g = g(x, \xi)$, which requires conditions ensuring that $(\partial_{x_i} g)(\cdot, u, \nabla u) \in L^p(\mathbb{R}^N)$. Sufficient conditions can be derived from Section 2 of [17]. The general Pohozaev identities are exactly as can be hinted from [15] by dropping the boundary integral. Likewise, similar identities are valid on more general unbounded domains Ω for solutions that vanish on $\partial\Omega$, but they must incorporate a boundary term (again suggested by [15]). The conditions required about Ω must ensure the exponential decay of the derivatives of the solutions up to order 2; see Remark 4.4. $\square$

6. Nonexistence of nonzero solutions in $W^{2,p}(\mathbb{R}^N)$.

The assumptions, namely (5.2) to (5.7), and the notation remain the same as in Section 5.

Theorem 6.1. *Retain assumptions (5.2) to (5.7). Let $p \in (N, \infty)$ and let $u \in W^{2,p}(\mathbb{R}^N)$ be a solution of (5.1). Suppose that for some real number a the condition*

$$(6.1) \qquad N[Q(\xi) + \Phi(\xi)] \geq (a+1) \sum_{\alpha=1}^{N} A_\alpha(\xi)\xi_\alpha + a[A_0(\xi) + \varphi(\xi_0)]\xi_0, \quad \forall\, \xi \in \mathbb{R}^{N+1},$$

holds. Then, $a \leq (N-2)/2$, and if for some $\xi_0^* > 0$ the set

$$(6.2) \qquad \{\xi_0 \in (-\xi_0^*, \xi_0^*) : \text{ equality holds in } (6.1) \text{ with } \xi = (\xi_0, \xi') \text{ and } \xi' \in \mathbb{R}^N \setminus \{0\}\},$$

has empty interior, then $u = 0$.

Proof. To see that $a \leq (N-2)/2$, write (6.1) for $\xi = (0, \xi')$ and $\xi' \in \mathbb{R}^N$. This yields $NQ_0(\xi') \geq (a+1)DQ_0(\xi')\xi'$, where $Q_0(\xi') := Q(0, \xi')$ (recall $A_\alpha = \partial_{\xi_\alpha} Q$). Since Q_0 is C^2 and $Q_0(0) = 0, DQ_0(0) = 0$ by (5.5), this inequality for small $|\xi'| > 0$ requires $a \leq (N-2)/2$ because of the ellipticity condition (5.3).

Now, multiply the first identity in Theorem 5.2 (ii) by a and add it to the second. It follows from (6.1) that $N[Q(u, \nabla u) + \Phi(u)] = (a+1) \sum_{\alpha=1}^{N} A_\alpha(u, \nabla u)\partial_\alpha u + a[A_0(u, \nabla u) + \varphi(u)]u$, i.e. equality holds in (6.1) for $\xi = (u(x), \nabla u(x))$ and every $x \in \mathbb{R}^N$.

By the continuity of u, the set $\Omega_0^* := u^{-1}((-\xi_0^*, \xi_0^*))$ is open in $\mathbb{R}^N$. We claim that $\nabla u = 0$ on Ω_0^*. Otherwise, let $x_0 \in \Omega_0^*$ be such that $\nabla u(x_0) \neq 0$. Since u is C^1, the implicit function theorem yields $\epsilon > 0$ and an open neighborhood $\mathcal{V}_{x_0}$ of x_0 in Ω_0^* such that $u(\mathcal{V}_{x_0}) = (u(x_0) - \epsilon, u(x_0) + \epsilon) \subset (-\xi_0^*, \xi_0^*)$ and $\nabla u(x) \neq 0$ for $x \in \mathcal{V}_{x_0}$. This shows that the set (6.2) contains $(u(x_0) - \epsilon, u(x_0) + \epsilon)$, in contradiction with the assumption that it has empty interior.

¿From the above, $\nabla u = 0$ on Ω_0^*, hence u is locally constant on Ω_0^*. Thus, u achieves only countably many values on Ω_0^*. Since $\Omega_0^* = u^{-1}((-\xi_0^*, \xi_0^*))$, this means that $u(\mathbb{R}^N) \cap (-\xi_0^*, \xi_0^*)$ is countable, and now the intermediate value theorem shows that $u(\mathbb{R}^N) \cap (-\xi_0^*, \xi_0^*)$ must be empty or reduce to one point. Since $\lim_{|x|\to\infty} u(x) = 0$, $u(\mathbb{R}^N) \cap (-\xi_0^*, \xi_0^*)$ is nonempty, and clearly the only point in it must be 0. In summary, u is a continuous function on $\mathbb{R}^N$ that achieves the value 0, but no other one, from $(-\xi_0^*, \xi_0^*)$. Obviously, the only such function is $u = 0$. $\square$

Remark 6.1: Write $\xi := (\xi_0, \xi')$ and, as in [15], assume that equality in (6.1) holds only when $\xi_0 = 0$ or $\xi' = 0$. Then, the set (6.2) reduces to $\{0\}$ irrespective of ξ_0^*. Thus, in this respect, Theorem 6.1 generalizes the condition given in [15]. This generalization will now allow us to rephrase Theorem 6.1 in a more convenient way. $\square$

For practical purposes, it is useful to split condition (6.1) into two inequalities. The way to do this is dictated by the remark that, by letting $\xi = (\xi_0, 0)$ in (6.1) and using (5.4) and (5.5), we obtain $N\Phi(\xi_0) \geq a\varphi(\xi_0)\xi_0$ for every $\xi_0 \in \mathbb{R}$. This (necesssary) inequality involves only the function Φ (since $\varphi = \Phi'$). Conversely,

Corollary 6.1. *Retain assumptions (5.2) to (5.7). Let $p \in (N, \infty)$ and let $u \in W^{2,p}(\mathbb{R}^N)$ be a solution of (5.1). Suppose that for some real number a with (necessarily) $a \leq (N-2)/2$, the conditions*

$$(6.3) \qquad NQ(\xi) \geq (a+1) \sum_{\alpha=1}^{N} A_\alpha(\xi)\xi_\alpha + aA_0(\xi)\xi_0, \quad \forall\, \xi \in \mathbb{R}^{N+1},$$

$$(6.4) \qquad N\Phi(\xi_0) \geq a\varphi(\xi_0)\xi_0, \qquad \forall\, \xi_0 \in \mathbb{R},$$

hold. Then $u = 0$.

Note: No assumption is needed about the set of points where equality holds in (6.3) or (6.4).

Proof. Since (6.3) and (6.4) imply (6.1), that $a \leq (N-2)/2$ was already seen in Theorem 6.1. To deduce $u = 0$ from that theorem, we show that there is $\xi_0^* > 0$ such that the inequality in (6.4) is strict for $0 < |\xi_0| < \xi_0^*$, for then the corresponding set (6.2) is $\{0\}$.

It follows from $\varphi'(0) > 0$ that $\varphi' > 0$ and Φ is convex on $(-\xi_0^*, \xi_0^*)$ for $\xi_0^* > 0$ small enough. Since $\Phi(0) = 0$, this implies $\Phi(\xi_0) \geq \varphi(\xi_0)\xi_0$ and hence $N\Phi(\xi_0) \geq N\varphi(\xi_0)\xi_0$ for $|\xi_0| < \xi_0^*$. Since φ is (strictly) increasing on $(-\xi_0^*, \xi_0^*)$ and $\varphi(0) = 0$, we have $\varphi(\xi_0)\xi_0 > 0$ for $0 < |\xi_0| < \xi_0^*$. Also, $a \leq (N-2)/2 < N$ implies $N - a > 0$. Thus, $N\Phi(\xi_0) \geq N\varphi(\xi_0)\xi_0 > a\varphi(\xi_0)\xi_0$ for $0 < |\xi_0| < \xi_0^*$. $\square$

Remark 6.2: When Q is independent of ξ_0, inequalitiies (6.3) and (6.4) are *equivalent* to (6.1). Also, in this case, (6.3) takes the form

$$(6.3') \qquad NQ(\xi') \geq (a+1) \sum_{\alpha=1}^{N} A_\alpha(\xi')\xi_\alpha, \quad \forall\, \xi' = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N. \quad \square$$

For the special problem (5.17), we have

Corollary 6.2. *Let $p \in (N/2, \infty), p > 1$, and let $u \in W^{2,p}(\mathbb{R}^N)$ be a solution of (5.17), where $\varphi(0) = 0$ and $\varphi'(0) > 0$. Suppose that*

$$(6.5) \qquad N\Phi(\xi_0) \geq a\varphi(\xi_0)\xi_0, \quad \forall\, \xi_0 \in \mathbb{R},$$

for some real number $a \leq (N-2)/2$. Then, $u = 0$.

Proof. In the proof of Theorem 5.3, we saw that every solution $u \in W^{2,p}(\mathbb{R}^N)$ of (5.17) is in $W^{2,q}(\mathbb{R}^N)$ for some $q \in (N, \infty)$. The conclusion now follows from Corollary 6.1 with $Q(\xi) = |\xi'|^2/2$, $\xi' = (\xi_1, \cdots, \xi_N)$ since (6.2) holds with any $a \leq (N-2)/2$ and (6.5) is the same as (6.4). $\square$

Corollary 6.2 can be inferred from the identities proved in Willem [21] or Kavian [11] for solutions $u \in H^1(\mathbb{R}^N)$, which is more general, but also under the additional assumptions that $\Phi(u) \in L^1(\mathbb{R}^N)$ and that equality in (6.5) occurs only when $\xi_0 = 0$. That $\Phi(u) \in L^1(\mathbb{R}^N)$ is here a consequence of Theorem 5.3.

Corollaries 6.1 and 6.2 can be complemented by looking at the case "$a = -\infty$", when only the first identity in Theorem 5.1 is used.

Theorem 6.2. *Retain assumptions (5.2) to (5.7). Let $p \in (N, \infty)$ and let $u \in W^{2,p}(\mathbb{R}^N)$ be a solution of (5.1). Suppose that*

$$\sum_{j=0}^{N} A_j(\xi)\xi_j \quad (= DQ(\xi)\xi) \geq 0, \quad \forall\, \xi \in \mathbb{R}^N, \tag{6.6}$$

$$\varphi(\xi_0)\xi_0 \geq 0, \quad \forall\, \xi_0 \in \mathbb{R}. \tag{6.7}$$

Then, $u = 0$.

Proof. Due to (5.7), the inequality (6.7) is strict for $0 < |\xi_0| < \xi_0^*$ and ξ_0^* small enough. Assuming $u \neq 0$, a contradiction is reached with the first identity in Theorem 5.1 (use the fact that $u^{-1}((-\xi_0^*, \xi_0^*) \setminus \{0\})$ is open and nonempty if $u \neq 0$). $\square$

We point out that in the $W^{2,p}(\mathbb{R}^N)$ setting (as opposed to the bounded domain case) Theorem 6.2 is not trivial. For the problem (5.17), condition (6.6) holds, whence only (6.7) has to be required (along with $\varphi'(0) > 0$) and Theorem 6.2 is valid for $p \in (N/2, \infty)$ by the argument of the proof of Corollary 6.2.

In contrast to the bounded domain case, other nonexistence results are obtained by reversing the inequalities in (6.1), (6.3) and (6.4), but they do not seem to incorporate simple or otherwise significant examples not already covered by Theorem 6.2. On the other hand, nonexistence results on more general unbounded domains can be derived whenever the Pohozaev identities remain available, as briefly discussed in Remark 5.2. The presence of a boundary term places further limitations to the geometry of the domain (star-shaped or star-shaped with respect to infinity) as explained in [15].

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